A WALSH NORMAL FORM FOR THE BINARY BURNSIDE OPERATOR, THE p^+-EXPANSION CONJECTURED BY DIACONIS, LIN AND RAM, AND THE SUPPORT BLOCKS OF THE k-ARY CHAIN

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Abstract. Let K_n be the transition matrix of the Burnside process on binary n-tuples under the coordinate action of S_n. Diaconis, Lin and Ram (arXiv:2512.23285) ask for K_n as an element of the universal enveloping algebra $U(\mathfrak{sl}_2)$ and conjecture, in their Conjecture 6.3, that K_n equals $\sum c_{y,z}\, f(x,y,z)$, summed over $x+y+z=n$, where $f(x,y,z)$ is the sum over all orderings of the Kronecker product of x copies of p^+, y of p^- and z of p^+h and the $c_{y,z}$ are explicit constants; they report the identity checked up to $n=10$ and do not pursue it. We prove it for every n. The proof turns on a normal form that does not appear in the source: with H the 2×2 Hadamard matrix, the transformed operator $2^{-n}H^{\otimes n}K_nH^{\otimes n}$ has (S,T) entry $c_{|S|,|T\setminus S|}$ when $S\subseteq T$ and 0 otherwise, so in the Walsh basis K_n is triangular for the inclusion order with an entry that does not depend on n. Three steps produce it: a cycle-parity lemma — for a uniform $\sigma\in S_m$ and a fixed t-set, the probability that every cycle of σ meets the set in an even number of points is $\binom{t}{t/2}2^{-t}$ for even t and 0 for odd t, independently of m — an alternating sum that yields the triangularity and removes n, and a constant-term evaluation that is von Szily's identity of 1894 for the super Catalan numbers. Consequently $c_{2p,2q}$ is the square of a normalised super Catalan number, the published eigenvalue table of the chain is a corollary of triangularity, and the source's closing remark that the constants $c_{k,k}$ are the eigenvalues β_k is corrected: $c_{1,1}=0\neq\frac14$, and the weights that are the eigenvalues are $c_{2k,0}=c_{0,2k}$. The normal form is verified by computer for $n\le 10$, and a reduction to the joint type of a pair of states carries a direct verification of the conjecture itself to every $n\le 24$.

The second half of the paper concerns the same chain on k-ary tuples and Conjecture 6.4 of the source, that for fixed k every nonzero eigenvalue of K_n has multiplicity $a_\lambda\binom{n}{b_\lambda}$. An integer change of basis adapted to the supports of words exhibits K_n, for every k and n, as block triangular with $\binom{n}{t}$ copies of one $(k-1)^t\times(k-1)^t$ matrix D_t on the level-t diagonal, so that $\operatorname{charpoly} K_n=\prod_t(\operatorname{charpoly} D_t)^{\binom{n}{t}}$ and the conjecture is equivalent to the nonzero spectra of $D_0,D_1,D_2,\dots$ being pairwise disjoint. For $k=3$ that disjointness is certified for all levels $t\le 16$ by explicit Bézout identities between the 62 integer polynomials that an exact factorisation, computed outside the proof assistant, identifies as the irreducible factors of the level characteristic polynomials, and for $t\le 8$ by exact annihilating identities $R_t(D_t)D_t=0$ that use no representation theory; the conjecture therefore holds for $k=3$ and every $n\le 16$. For $k=4$ it holds for every $n\le 5$. The eigenvalue $\frac{1}{18}$ the source singles out, with multiplicities $2,10,30,70$ at $n=4,\dots,7$, is pinned to the unique level 4 with two explicit integral eigenvectors, and it is the first member θ_2 of a rectangular family θ_a with $\theta_4=\frac{4}{243}$, $\theta_6=\frac{6857}{885735}$, $\theta_8=\frac{357161}{79716150}$ and multiplicity $\operatorname{Cat}(a)\binom{n}{2a}$. Every finite statement made as a theorem here, and the arithmetic of the general proof of Conjecture 6.3 for all parameters, is machine-checked in the Lean 4 proof assistant; the two general block-decomposition arguments and the Schur–Weyl bookkeeping are ordinary mathematics and are not.

1. Introduction

1.1. **The Burnside process.** Let a finite group G act on a finite set $\mathfrak{X}$. One step of the *Burnside process* from $x\in\mathfrak{X}$ picks s uniformly from the stabiliser G_x and then picks y uniformly from the fixed-point set $\mathfrak{X}_s=\{y : y^s=y\}$. Its transition matrix is

$$K(x,y)=\frac{1}{|G_x|}\sum_{s\in G_x\cap G_y}\frac{1}{|\mathfrak{X}_s|}, \tag{1}$$

and the chain is reversible with stationary distribution proportional to $1/|\mathcal{O}_x|$, $\mathcal{O}_x$ the orbit of x; recording only the orbit therefore samples orbits uniformly, which is what the process is for [4, 5].

The smallest nontrivial instance, and the one studied in the first half of this paper, is $\mathfrak{X} = C_2^n$, the binary n-tuples, with $G = S_n$ permuting coordinates. Then $G_x = S_{|x|} \times S_{n-|x|}$, where $|x|$ is the number of ones, and $|\mathfrak{X}_s| = 2^{c(s)}$ with $c(s)$ the number of disjoint cycles of s (fixed points included), so (1) reads

$$K(x,y) = \frac{1}{|G_x|} \sum_{s \in G_x \cap G_y} 2^{-c(s)}. \tag{2}$$

Concretely: from x choose $s \in S_n$ with $x^s = x$ uniformly, break s into disjoint cycles, colour each cycle 0 or 1 with probability $\frac{1}{2}$, and install the values along the cycles. Write K_n for the resulting $2^n \times 2^n$ matrix, viewed as an operator on $V^{\otimes n}$ with $V = \mathbb{Q}^2$ spanned by v_0, v_1. Its spectrum is known [2, Theorem 1.2]: the eigenvalues are

$$\beta_k = \frac{\binom{2k}{k}^2}{2^{4k}} \quad \text{with multiplicity } \binom{n}{2k} \ (0 \le k \le \lfloor n/2 \rfloor), \qquad \text{and } 0 \text{ with multiplicity } 2^{n-1}. \tag{3}$$

[1] quotes the list in this form, as its equation (1.1). The same process on k-ary tuples $\mathfrak{X} = C_k^n = (\mathbb{Z}/k)^n$ — label each cycle of s uniformly from $\mathbb{Z}/k$, so that $|\mathfrak{X}_s| = k^{c(s)}$ — is the subject of §8.

1.2. **The conjecture.** Because K_n commutes with the S_n-action, Schur–Weyl duality makes it the action of some element of the universal enveloping algebra $U(\mathfrak{sl}_2)$ on $V^{\otimes n}$; and because the nonzero eigenvalues (3) do not depend on n, [1] raises the question whether *one* element of $U(\mathfrak{sl}_2)$ works for every n. It introduces its Conjecture 6.3 with the words

> "The following conjecture (which has been checked up to $n = 10$) is one direction in which this idea could be further explored (though we do not do so here)"

and states, with e, f, h the standard basis of $\mathfrak{sl}_2$,

$$p^+ = \tfrac{1}{2}(1+e+f) = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad p^- = \tfrac{1}{2}(1-e-f) = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad p^+h = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \tag{4}$$

and, letting $f(x,y,z)$ be "the sum over all ways (orders) of taking the matrix tensor product of x copies of p^+, y copies of p^-, and z copies of p^+h",

$$K_n = \sum_{x+y+z=n} c_{y,z}\, f(x,y,z), \qquad c_{y,z} = \begin{cases} \left(\dfrac{y!\, z!}{\left(\frac{y}{2}\right)!\left(\frac{z}{2}\right)!\left(\frac{y+z}{2}\right)!\, 2^{y+z}} \right)^2, & y, z \text{ nonnegative even}, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

(The letter f carries two meanings in (4)–(5); both are the source's, and we keep them, the ordering sum being always written with its three arguments.) The paper prints the expansions of K_4 and K_6 and closes the discussion with a paragraph of two remarks, the first of which we correct in §6:

> "Also note that the constants $c_{k,k}$ are exactly the eigenvalues β_k of our Markov chain. It may be interesting to write out more explicit expressions for the various terms $f(x,y,z)$, or to find probabilistic interpretations for the off-diagonal constants $c_{y,z}$…"

The same conjecture stood in the first version of the companion paper [2], whose second version moved the Schur–Weyl material into [1].

The last conjecture of [1], its Conjecture 6.4, which it attributes to [2, Conjecture 6.2], concerns the k-ary chain:

> "Fix k, and let λ be any nonzero eigenvalue of the Burnside chain on (C_k^n, S_n) for any n. Then λ occurs with multiplicity $a_\lambda \binom{n}{b_\lambda}$ for some integers a_λ, b_λ."

For $k = 2$ this is (3). The evidence [2] offers for $k = 3$ is one eigenvalue: in the paragraph following its Conjecture 6.2,

> "In contrast, consider the eigenvalue $\frac{1}{18}$ for $k = 3$. This eigenvalue does not appear in the orbit Bose-Einstein chain for any $n \le 8$, but it occurs with multiplicity $2, 10, 30, 70$ for $n = 4, 5, 6, 7$, suggesting that $a_\lambda = 2$ and $b_\lambda = 4$."

1.3. **Results.** *(i) The conjecture is a theorem.* Theorem 5.1 proves (5) for every $n \geq 0$. The argument is elementary once the basis is right, and it is given in full in §5.

(ii) A normal form. Conjugating by the Hadamard matrix $H = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ sends p^+, p^-, p^+h to the matrix units E_{11}, E_{22}, E_{12} (Lemma 3.1), so a word in the three letters becomes a single matrix unit $E_{S,T}$ indexed by a pair $S \subseteq T$ of subsets of $\{1, \dots, n\}$, and the conjecture becomes a statement about the Walsh transform $\widetilde{K}_n = 2^{-n} H^{\otimes n} K_n H^{\otimes n}$:

$$\widetilde{K}_n[S,T] = \begin{cases} c_{|S|,\,|T\setminus S|}, & S \subseteq T, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

That is Theorem 3.3: in the Walsh basis the Burnside operator is triangular for the inclusion order, and its entries do not depend on n at all. Neither the transform nor the triangularity occurs in [1], whose own eigenvectors $f_S = \sum_T (-1)^{|S\cap T|} \binom{|S|}{|S\cap T|} v_T$ carry an extra binomial weight and span a different basis; the full text of [1] contains no Walsh, Hadamard or Fourier transform. Statement (6) is what makes the conjecture provable for all n, and it is verified directly by computer for $n \leq 10$.

(iii) Two ingredients of independent interest. The cycle-parity lemma (Lemma 4.1): for a uniform $\sigma \in S_m$ and a fixed t-subset, the probability that every cycle of σ meets the subset in an even number of points equals $\gamma_t = \binom{t}{t/2} 2^{-t}$ for even t and 0 for odd t — *independently of* m. And the core identity (Lemma 4.5), whose arithmetic content is von Szily's identity of 1894 for the super Catalan numbers $\mathrm{sc}(p,q) = (2p)!(2q)!/(p!\,q!\,(p+q)!)$.

(iv) What the constants are. Proposition 4.6: $c_{y,z} = c_{z,y}$, the weight vanishes as soon as one argument is odd, and $c_{2p,2q} = \left(\mathrm{sc}(p,q)/4^{p+q}\right)^2$. So the constants of (5) are squares of normalised super Catalan numbers. This is an arithmetic identification, not the probabilistic interpretation [1] asks for, but it is the reason von Szily's identity is the arithmetic core of the proof.

(v) A correction. The remark quoted above misidentifies the constants: $c_{k,k} = 0$ for odd k, so the identification fails already at $k = 1$ ($c_{1,1} = 0 \neq \frac{1}{4} = \beta_1$), and $c_{2,2} = \frac{1}{64} \neq \frac{9}{64} = \beta_2$. The weights that are the eigenvalues are $c_{2k,0} = c_{0,2k}$ (Proposition 6.2), which is what the diagonal of (6) predicts. Nothing in [1] depends on the remark.

(vi) Direct verification, further than before. Both sides of (5) depend on a pair of states only through its joint type, a quadruple of nonnegative integers summing to n. Theorem 7.2 records the resulting identity, checked for every one of the $\binom{n+3}{3}$ types at every $n \leq 24$, against the $n \leq 10$ reported in the source.

(vii) The arithmetic of the general proof, machine-checked for all parameters. The three arithmetic steps of the proof of Theorem 5.1 — von Szily's identity (Lemma 4.4), the core identity (Lemma 4.5) and the cycle-parity constant in the closed-form language of Proposition 2.2 (Lemma 4.3) — and the triangularity-and-n-freeness step at the level of the four-region sum (Theorem 5.2) are now proved in the proof assistant for *all* values of their parameters, by symbolic proofs with no compiled evaluation. What is not machine-checked is the translation between these sums and the matrices, so Theorem 5.1 itself remains an ordinary proof; §9 draws the line exactly.

(viii) The k-ary chain: a block decomposition. Section 8 turns to Conjecture 6.4. On functions on $\mathbb{Z}/k$ take the basis $g_0 = 1$, $g_r = \delta_r$ ($r = 1, \dots, k-1$); it is unimodular over $\mathbb{Z}$ and adapted to supports, and in the basis $g_{a_1} \otimes \cdots \otimes g_{a_n}$ the operator K_n is triangular for inclusion of supports with an n-free entry (Proposition 8.3; checked exactly, entry by entry, in the ranges of Theorem 8.4): the entry at (a', a) vanishes unless $\operatorname{supp} a' \subseteq \operatorname{supp} a$, and when $|\operatorname{supp} a| = t$ it equals the corresponding entry of the same normal form at $n = t$. Consequently, with D_t the $(k-1)^t \times (k-1)^t$ block of full support at $n = t$,

$$\operatorname{charpoly} K_n = \prod_{t=0}^{n} (\operatorname{charpoly} D_t)^{\binom{n}{t}} \quad \text{for every } k \text{ and } n,$$

and Conjecture 6.4 for a given k is exactly the statement that the nonzero spectra of $D_0, D_1, D_2, \dots$ are pairwise disjoint (Corollary 8.5). For $k = 2$ every D_t is 1×1 and equals $c_{t,0}$, which recovers (3); for $k \geq 3$ the scalar identity of (6) is replaced by a matrix, and there is no super Catalan formula to look for.

(ix) Conjecture 6.4 for $k = 3$ and $n \le 16$, and for $k = 4$ and $n \le 5$. Theorem 8.6: an exact factorisation over $\mathbb{Q}$, computed outside the proof assistant, splits the nonzero parts of the characteristic polynomials of $D_0, \dots, D_{16}$ for $k = 3$ into 62 irreducible polynomials of degree at most 3, and for every one of the 1891 pairs an explicit Bézout identity $uP_i + vP_j = c \in \mathbb{Z} \setminus \{0\}$ is exhibited and checked inside the proof assistant, so no two levels share a nonzero eigenvalue. Theorem 8.7: for $t \le 8$ the product R_t of the level-t factors satisfies $R_t(D_t)D_t = 0$ exactly, which locates the spectrum of D_t inside the roots of R_t with no representation theory, so that for $n \le 8$ the conclusion rests on nothing that is cited. Theorem 8.10 does the same for $k = 4$ (coprimality of the level factors for $t \le 6$, annihilators for $t \le 5$), giving the conjecture for every $n \le 5$; this is the first exact statement about the conjecture for any $k \ge 4$.

(x) The eigenvalue $\frac{1}{18}$, and a rectangular family. Theorem 8.8: $\frac{1}{18}$ is an eigenvalue of D_4 ($k = 3$) with two explicit integral eigenvectors, and of no other D_t with $t \le 8$; so $b_{1/18} = 4$ is forced, not suggested, and the source's $2, 10, 30, 70$ are $2\binom{n}{4}$. Theorem 8.9: for $\mu = (a^{k-1})$ the $S_t \times GL_{k-1}$ decomposition of D_t at $t = a(k-1)$ contains a one-dimensional block, on which D_t is a scalar θ_a computable from the n-free type formula; for $k = 3$, $\theta_2 = \frac{1}{18}$, $\theta_4 = \frac{4}{243}$, $\theta_6 = \frac{6857}{885735}$, $\theta_8 = \frac{357161}{79716150}$, $\theta_a = 0$ for odd $a \le 7$, and $\theta_2 = \frac{3}{256}$ for $k = 4$, $\theta_2 = \frac{3}{1250}$ for $k = 5$.

1.4. **What is not claimed.** We do not exhibit the element of $U(\mathfrak{sl}_2)$ that motivates [1, §6]: (6) answers the neighbouring question of what K_n is in the Walsh basis, and the triangular n-free form is a natural input to the original question, but it is not an answer to it. We give no probabilistic interpretation of the off-diagonal constants. Conjecture 6.4 is not proved for any $k \ge 3$: it is reduced to an equivalent statement about the finite matrices D_t and certified in the ranges stated in (ix), and §8.8 records why the data suggest no short structural proof. The identification of the 62 polynomials of Theorem 8.6 with the factors of $\operatorname{charpoly} D_t$ for $9 \le t \le 16$ rests on an exact factorisation computed outside the proof assistant through Schur–Weyl duality, and the multiplicity $\mathrm{Cat}(a)$ of θ_a rests on the same duality; both are said where they are used. Finally, Theorem 5.1 and Proposition 8.3 are ordinary mathematical proofs and are not machine-checked, and neither are the shorter general arguments that connect the finite checks to them (Propositions 3.2 and 7.1, Corollaries 6.1 and 8.5, the Schur–Weyl decomposition of D_t); §9 says exactly what is.

1.5. **Changes from version 1.** Version 1 of this paper contained (i)–(vi). This version adds: the all-parameter machine-checked forms of von Szily's identity, the core identity, the cycle-parity constant and the region sum (Lemma 4.4, the second proof of Lemma 4.5, Lemma 4.3, Theorem 5.2); the whole of §8 on the k-ary chain and Conjecture 6.4; a rewritten §9; and one reference [3]. Nothing stated in version 1 is withdrawn or weakened; the sentence of its introduction that nothing in the paper bears on Conjecture 6.4 is superseded by §8, and the second proof of Lemma 4.5 replaces a two-variable Laurent-series substitution by a coefficient extraction in $\mathbb{Z}[u][v]$.

2. The objects

Throughout, $[n] = \{1, \dots, n\}$, and a state $x \in C_2^n$ is identified with its support $\{i : x_i = 1\} \subseteq [n]$; $|x|$ is the number of ones. For subsets $S, T \subseteq [n]$ we write $|S \cap T|$ for the size of the intersection, and $v_T = v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n}$, with $\epsilon_i = 1$ for $i \in T$ and $\epsilon_i = 0$ otherwise, for the standard basis vector of $V^{\otimes n}$ indexed by T.

Definition 2.1. K_n is the $2^n \times 2^n$ matrix (2). For $t \ge 0$ put

$$\gamma_t = \begin{cases} \binom{t}{t/2} 2^{-t}, & t \text{ even}, \\ 0, & t \text{ odd}, \end{cases} \qquad R(m) = \sum_{\sigma \in S_m} 2^{-c(\sigma)}, \qquad w(s,t) = \frac{R(s)R(t)}{(s+t)!},$$

and let $c_{y,z}$ be the weight of (5), β_k the eigenvalue of (3), and $\mathrm{sc}(p,q) = (2p)!(2q)!/(p!\,q!\,(p+q)!)$ the super Catalan number. The Hadamard matrix is $H = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, whose columns are $v_0 + v_1$ and $v_0 - v_1$; since $H^2 = 2I$, the *Walsh transform* of an operator M on $V^{\otimes n}$ is

$$\widetilde{M} = H^{-\otimes n} M H^{\otimes n} = 2^{-n} H^{\otimes n} M H^{\otimes n}, \qquad \text{entrywise} \quad \widetilde{M}[S,T] = 2^{-n} \sum_{x,y} (-1)^{|S \cap x| + |T \cap y|} M[x,y].$$

Only the symbols K_n, $p^{\pm}$, p^+h, $f(x,y,z)$, $c_{y,z}$ and β_k are the source's; γ_t, R, w, sc and the transform are ours, introduced to keep the statements short.

Two conventions. First, $c_{y,z}$ is defined by (5) for *all* pairs of nonnegative integers, not only those occurring in a given n; every statement below about c is a statement about that function. Second, the ordering sum $f(x,y,z)$ ranges over all $n!/(x!\,y!\,z!)$ words in the three letters, so that the right-hand side of (5) is a sum over all 3^n words $w \in \{p^+, p^-, p^+h\}^n$ of $c_{y(w),z(w)}\, w_1 \otimes \cdots \otimes w_n$, where $y(w)$ and $z(w)$ count the letters p^- and p^+h in w. Replacing the sum over orderings by the multinomial coefficient times the sorted word is a different statement, and a false one (Proposition 7.4).

The kernel (2) has a closed form, which we record here because it is what makes the computations of §3 and §7 affordable. Call

$$\tau(x,y) = (a_0, a_1, b_0, b_1), \qquad a_c = \#\{i : x_i = 0,\ y_i = c\}, \quad b_c = \#\{i : x_i = 1,\ y_i = c\},$$

the *joint type* of the pair (x,y); it is a quadruple of nonnegative integers summing to n.

Proposition 2.2. *$R(m) = (2m)!/(4^m m!)$ for every $m \geq 0$, and*

$$K(x,y) = w(a_0,a_1)\, w(b_0,b_1), \qquad (a_0,a_1,b_0,b_1) = \tau(x,y).$$

In particular $K(x,y)$ depends on (x,y) only through the joint type.

Proof. The classical evaluation $\sum_{\sigma \in S_m} u^{c(\sigma)} = u(u+1)\cdots(u+m-1)$ at $u = \frac{1}{2}$ gives $R(m) = \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2m-1}{2} = (2m-1)!!\, 2^{-m} = (2m)!/(4^m m!)$. For the factorisation, write $A_c = \{i : x_i = 0, y_i = c\}$ and $B_c = \{i : x_i = 1, y_i = c\}$. A permutation $s \in S_n$ lies in $G_x \cap G_y$ exactly when it preserves both x and y, i.e. when it preserves each of the four sets, so $G_x \cap G_y = S_{A_0} \times S_{A_1} \times S_{B_0} \times S_{B_1}$ and $c(s)$ is the sum of the four cycle counts. Hence the sum in (2) is $R(a_0)R(a_1)R(b_0)R(b_1)$, and $|G_x| = (a_0+a_1)!\,(b_0+b_1)!$. □

Both halves were also checked by machine: $R(m)$ against the enumeration of S_m for $m \leq 7$, and the closed form against the matrix built literally from (2) for $n \leq 6$.

3. The Hadamard basis and the Walsh normal form

3.1. **Three matrix units.** The following identities are three 2×2 multiplications; we state them after clearing the common factor 2, so that they are identities between integer matrices.

Lemma 3.1. *With e, f, h as in (4) one has $2p^+ = 1 + e + f$, $2p^- = 1 - e - f$, $2p^+h = (1+e+f)h$, the relations $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$, and p^+, p^- are complementary orthogonal idempotents with $(p^+h)^2 = 0$. Moreover*

$$p^+ H = H E_{11}, \qquad p^- H = H E_{22}, \qquad p^+h\, H = H E_{12},$$

so that, for a word $w \in \{p^+, p^-, p^+h\}^n$,

$$H^{-\otimes n}\big(w_1 \otimes \cdots \otimes w_n\big) H^{\otimes n} = E_{S,T}, \qquad S = \{i : w_i = p^-\}, \quad T = S \cup \{i : w_i = p^+h\},$$

and $w \mapsto (S,T)$ is a bijection from words onto pairs $S \subseteq T \subseteq [n]$, with $|S| = y(w)$ and $|T \setminus S| = z(w)$.

Proof. The displayed products are immediate from (4); for instance $p^+H = \frac{1}{2}\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\1&-1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right) = HE_{11}$. Conjugation is multiplicative for Kronecker products, and $E_{i_1 j_1} \otimes \cdots \otimes E_{i_n j_n}$ is the matrix unit $E_{S,T}$ whose row index is the state with a 1 exactly where $i_k = 2$ and whose column index is the state with a 1 exactly where $j_k = 2$; the three letters contribute $(i,j) = (1,1), (2,2), (1,2)$, so the row index is S, the column index is $T \supseteq S$, and every pair $S \subseteq T$ arises from exactly one word. □

Lemma 3.1 converts (5) into a statement with no orderings in it.

Proposition 3.2. *For each n, Conjecture 6.3 of* [1] *at n — that is,* (5) *— holds if and only if the Walsh transform $\widetilde{K}_n$ satisfies* (6)*.*

Proof. By Lemma 3.1 the right-hand side of (5) is $\sum_w c_{y(w),z(w)}\, w_1 \otimes \cdots \otimes w_n$, whose transform is $\sum_{S \subseteq T} c_{|S|,|T\setminus S|} E_{S,T}$, the matrix on the right of (6). The transform is a linear isomorphism, so the two identities are equivalent. □

3.2. **The normal form.**

Theorem 3.3. *For $1 \le n \le 7$, with K_n built directly from the two-step description* (2)*, the Walsh transform satisfies*

$$\widetilde{K}_n[S,T] = c_{|S|,\,|T\setminus S|} \quad (S \subseteq T), \qquad \widetilde{K}_n[S,T] = 0 \quad (S \not\subseteq T).$$

The same holds for $8 \le n \le 10$ with K_n evaluated through the closed form of Proposition 2.2. The right-hand side of (5) *has the same transform for $1 \le n \le 5$.*

Proof. Each assertion is a finite identity between arrays of exact rationals, decided by exhaustive computation: for $n \le 7$ the matrix K_n is assembled by walking all $n!$ permutations and distributing $2^{-c(s)}/|G_x|$ over the pairs of states each fixes, and its transform is computed by a fast Walsh–Hadamard pass over rows and then over columns; for $8 \le n \le 10$ the entries are taken from Proposition 2.2 instead, which removes the permutation enumeration and is what makes those three cases affordable. The largest computation is $n = 10$, a 1024×1024 rational matrix and its transform. For every n the statement is Theorem 5.1 below, of which these are instances. □

The transform is triangular for a partial order, and the entry at (S,T) depends on (S,T) only through the two numbers $|S|$ and $|T \setminus S|$; in particular it does not depend on n, which is the point. The computation is not vacuous in the ways one would want to exclude. The right-hand side of (6) is not diagonal — its entry at $(\emptyset, \{1,2\})$ is $c_{0,2} = \frac{1}{4}$, while its entry at $(\{1,2\}, \emptyset)$ is 0 — and it is not symmetric; the transform is an involution, so (6) really is equivalent to a statement about K_n itself; and replacing $c_{2,0}$ by $2c_{2,0}$ destroys (6) already at $n = 2$ and again at $n = 3$. We also record that the source's own printed coefficients are the ones (5) produces.

Proposition 3.4. *$c_{0,0} = 1$, $c_{2,0} = c_{0,2} = \frac{1}{4}$, $c_{4,0} = c_{0,4} = \frac{9}{64}$, $c_{2,2} = \frac{1}{64}$, $c_{6,0} = c_{0,6} = \frac{25}{256}$, $c_{2,4} = c_{4,2} = \frac{1}{256}$ — the weights appearing in the K_4 and K_6 expansions printed in* [1] *— and $c_{y,z} = 0$ exactly when y or z is odd, for all $y + z \le 8$.*

4. Two lemmas, and what the constants are

4.1. **Cycle parity.** The first lemma is the probabilistic content of the whole argument. Both proofs below are routine; what matters is the conclusion, that the answer does not depend on m.

Lemma 4.1. *Fix $m \ge 0$, a subset $M \subseteq [m]$ with $|M| = t$, and let $\sigma \in S_m$ be uniform. Then*

$$q(m,t) := \mathbb{P}\big(\textit{every cycle of } \sigma \textit{ meets } M \textit{ in an even number of points}\big) = \gamma_t,$$

independently of m.

Proof. Mark the points of M and leave the others unmarked. A cycle of length r carrying an even number of marked points has bivariate exponential generating function $\frac{1}{r} \cdot \frac{1}{2}\big((u+v)^r + (v-u)^r\big)$, u marking marked points and v unmarked ones, so summing over $r \ge 1$ and exponentiating, permutations all of whose cycles are even-marked have exponential generating function

$$F(u,v) = \exp\Big(-\tfrac{1}{2}\log\big((1-u-v)(1-v+u)\big)\Big) = \big((1-v)^2 - u^2\big)^{-1/2} = \sum_{k \ge 0} \binom{2k}{k} 4^{-k} u^{2k} (1-v)^{-2k-1}.$$

Extracting $[u^t v^{m-t}]$ gives 0 for odd t and $\binom{2k}{k}4^{-k}\binom{m}{2k}$ for $t = 2k$; multiplying by $t!\,(m-t)!$ counts the permutations in question and dividing by $m!$ gives $q(m,t) = \binom{2k}{k}4^{-k}\binom{m}{2k}/\binom{m}{t} = \binom{t}{t/2}2^{-t}$. □

Remark 4.2. A bijective proof is available and explains the m-independence better: deleting an unmarked point from its cycle is an $(m+1)$-to-1 map $S_{m+1} \to S_m$ preserving the property, so $q(m,t) = q(t,t)$, and $q(t,t)$ is the proportion of permutations of $[t]$ all of whose cycles have even length, namely $((t-1)!!)^2/t! = \binom{t}{t/2}2^{-t}$ for even t and 0 for odd t. The identity $q(m,t) = \gamma_t$ was also confirmed by exhaustive enumeration for all $m \le 8$ and all $t \le m$.

The lemma enters the proof of Theorem 5.1 through the closed form of Proposition 2.2, and in that language it reads as follows. Fix a block of size m carrying t marked points, and sum the closed-form weight $w(m-|u|,|u|)$ over all subsets u of the block with the sign $(-1)^{|u\cap M|}$; grouping the subsets by $|u|=i$, the signed count of those with $|u|=i$ is the coefficient of X^i in $(1-X)^t(1+X)^{m-t}$.

Lemma 4.3 (cycle parity, closed form)**.** *Let $R(m)=(2m)!/(4^m m!)$ and $w(s,t)=R(s)R(t)/(s+t)!$. For all $0\le t\le m$,*

$$\sum_{i=0}^{m}\big[X^i\big]\big((1-X)^t(1+X)^{m-t}\big)\,w(m-i,i)=\gamma_t,$$

independently of m.

Proof. Write $R(i)=(2i-1)!!\,2^{-i}$ and clear denominators: the claim is $L(m,t)=2^{m-t}m!\binom{t}{t/2}$ for even t and $L(m,t)=0$ for odd t, where $L(m,t)=\sum_i A_i\,(2i-1)!!\,(2(m-i)-1)!!$ and $A_i=[X^i](1-X)^t(1+X)^{m-t}$. Pascal's rule for the A_i together with $(2i+1)!!=(2i+1)(2i-1)!!$ gives, in one line, $L(m+1,t)=2(m+1)L(m,t)$ — the two brackets $2(m-i)+1$ and $2i+1$ add to $2m+2$, with no i left — so everything reduces to the diagonal $L(t,t)$. At $t=0$ the recursion is the central-binomial convolution $\sum_i\binom{2i}{i}\binom{2(m-i)}{m-i}=4^m$ (after $\binom{2i}{i}i!=2^i(2i-1)!!$). On the diagonal $A_i=(-1)^i\binom{t}{i}$, and the same substitution gives $L(t,t)=2^{-t}t!\sum_i(-1)^i\binom{2i}{i}\binom{2(t-i)}{t-i}$, so the claim $L(t,t)=t!\,\mathrm{mid}(t)$ is the alternating convolution

$$\sum_{i=0}^{t}(-1)^i\binom{2i}{i}\binom{2(t-i)}{t-i}=\begin{cases}2^t\binom{t}{t/2}, & t\text{ even},\\ 0, & t\text{ odd},\end{cases}$$

which is proved by square-root uniqueness in $\mathbb{Z}[[X]]$: the series $P(-X)P(X)$ with $P=\sum_i\binom{2i}{i}X^i$ and the series $\sum_t 2^t\binom{t}{t/2}X^t$ (even t only) have the same square, namely $\sum_n(-4)^nX^n\cdot\sum_n 4^nX^n$ by the plain convolution, and the same constant term 1; and two integer sequences with equal convolution squares and constant term 1 are equal, by induction on the index. □

Through $R(m)=\sum_{\sigma\in S_m}2^{-c(\sigma)}$ (Proposition 2.2) the left-hand side is $q(m,t)$, so Lemma 4.3 is Lemma 4.1 in the closed-form language; the two proofs above make no use of it. The distinction matters for §9: Lemma 4.3 is machine-checked for all m and t as stated, and nothing about permutations is machine-checked beyond the finite instances ($R(m)$ against enumeration for $m\le 7$, $q(m,t)$ for $m\le 8$). The alternating convolution takes the values $1,8,96,1280,17920$ at $t=0,2,4,6,8$ (OEIS A098430); the sign is doing work, since at $t=1$ the alternating and the plain convolutions are 0 and 4.

4.2. **The core identity.** For $y,z\ge 0$ write $N=y+z$ and

$$\Theta(y,z)=\sum_{i=0}^{y}\sum_{j=0}^{z}(-1)^i\binom{y}{i}\binom{z}{j}\gamma_{i+j}\,\gamma_{N-i-j}. \tag{7}$$

The evaluation of Θ is where the constants of (5) come from, and it rests on a classical identity of von Szily.

Lemma 4.4 (von Szily [6])**.** *For all $p,q\ge 0$, the sum over $k\in\mathbb{Z}$ of finitely many nonzero terms,*

$$\sum_k(-1)^k\binom{2p}{p+k}\binom{2q}{q+k}=\mathrm{sc}(p,q)=\frac{(2p)!(2q)!}{p!\,q!\,(p+q)!};$$

equivalently, in cleared-denominator form, $p!\,q!\,(p+q)!\sum_i(-1)^i\binom{2p}{i}\binom{2q}{p+q-i}=(-1)^p(2p)!(2q)!$. In particular $\mathrm{sc}(p,q)$ is an integer. Moreover, for $A_{a,b}(X)=(1-X)^a(1+X)^b\in\mathbb{Z}[X]$ one has $X^{a+b}A_{a,b}(1/X)=(-1)^aA_{a,b}(X)$, so the middle coefficient $[X^h]A_{a,b}$ vanishes whenever $a+b=2h$ and a is odd.

Proof. The alternating sum is $[X^{p+q}]A_{2p,2q}$. Differentiating, $(1-X^2)A'_{a,b}=A_{a,b}\big((b-a)-(a+b)X\big)$, and comparing coefficients gives the three-term recurrence $(m+2)A_{m+2}=(b-a)A_{m+1}+(m-a-b)A_m$ for the coefficients $A_m=[X^m]A_{a,b}$. With $a=2p$, $b=2q$ and $F(p,q)=[X^{p+q}]A_{2p,2q}$, writing $(1-X)^{2p+2}=(1-2X+X^2)(1-X)^{2p}$ and applying the recurrence at $m=p+q-1$ makes the A_{p+q-1} terms cancel

identically and leaves $(p+q+1)F(p+1,q) = -2(2p+1)F(p,q)$. Induction on p from $F(0,q) = \binom{2q}{q}$ gives $F(p,q)\,p!\,q!\,(p+q)! = (-1)^p(2p)!(2q)!$, which is the cleared-denominator form; the substitution $i = p+k$ turns it into the first form. Integrality of $\mathrm{sc}(p,q)$ follows since the sum is an integer. The palindromy is $(1-1/X)^a(1+1/X)^b = X^{-a-b}(X-1)^a(X+1)^b$, and comparing the coefficient of X^h on both sides gives $A_h = (-1)^a A_h$. □

The identity is classical; at $p=q$ it reads $\sum_k (-1)^k \binom{2p}{p+k}^2 = \binom{2p}{p}$, an alternating sum of squares of binomial coefficients. For the super Catalan numbers see also [7]. The proof above is the one that is machine-checked, for all p,q (the sign is not vacuous: without $(-1)^p$ the identity already fails at $p=1$, $q=0$, and $[X^2]A_{2,2} = -2$, so the parity hypothesis of the last assertion is doing work); the cleared-denominator form was in addition re-verified by exact integer arithmetic for all $p,q \le 10$.

Lemma 4.5. *For all $y,z \ge 0$ one has $\Theta(y,z) = 2^{\,y+z} c_{y,z}$.*

Proof. Work in $\mathbb{Q}[w^{\pm1}, w'^{\pm1}]$ and let CT denote the constant term. Since $\gamma_s = \mathrm{CT}_w\big[\big(\frac{w+w^{-1}}{2}\big)^s\big]$, writing $\mathcal{A} = \frac12(w+w^{-1})$ and $\mathcal{B} = \frac12(w'+w'^{-1})$ and using the binomial theorem twice,

$$\Theta(y,z) = \mathrm{CT}_{w,w'}\Big[\sum_{i,j}(-1)^i\binom{y}{i}\binom{z}{j}\mathcal{A}^{i+j}\mathcal{B}^{N-i-j}\Big] = \mathrm{CT}_{w,w'}\big[(\mathcal{B}-\mathcal{A})^y(\mathcal{B}+\mathcal{A})^z\big].$$

Substitute $w = r/s$, $w' = rs$. The monomial $w^a w'^b$ becomes $r^{a+b}s^{b-a}$, and $(a,b) \mapsto (a+b, b-a)$ is injective with $(0,0)$ its only preimage of $(0,0)$, so constant terms are unchanged. Moreover

$$\mathcal{B}-\mathcal{A} = \tfrac12(r-r^{-1})(s-s^{-1}), \qquad \mathcal{B}+\mathcal{A} = \tfrac12(r+r^{-1})(s+s^{-1}),$$

whence, with $D(t) = (t-t^{-1})^y(t+t^{-1})^z$, the constant term factorises:

$$\Theta(y,z) = 2^{-N}\big(\mathrm{CT}_t[D]\big)^2.$$

Now $D(t) = t^N(1-u)^y(1+u)^z$ with $u = t^{-2}$, so $\mathrm{CT}_t[D] = [u^{N/2}](1-u)^y(1+u)^z$, which vanishes unless N is even. The constant term is invariant under $t \mapsto -t$ and under $t \mapsto -1/t$ (both fix the coefficient of t^0), while D is multiplied by $(-1)^N$ by the first and by $(-1)^z$ by the second; hence $\mathrm{CT}_t[D] = 0$ unless y and z are both even. For $y = 2p$, $z = 2q$ the substitution $i = p+k$ turns $[u^{p+q}](1-u)^{2p}(1+u)^{2q} = \sum_i (-1)^i\binom{2p}{i}\binom{2q}{p+q-i}$ into $(-1)^p\sum_k(-1)^k\binom{2p}{p+k}\binom{2q}{q+k}$, so by Lemma 4.4

$$\mathrm{CT}_t[D] = (-1)^{y/2}\,\frac{y!\,z!}{\big(\frac{y}{2}\big)!\,\big(\frac{z}{2}\big)!\,\big(\frac{y+z}{2}\big)!}.$$

Squaring and comparing with (5), $\Theta(y,z) = 2^{-N}\cdot 2^{2N}c_{y,z} = 2^N c_{y,z}$, and both sides vanish when y or z is odd. □

Second proof. Put $\mathrm{mid}(t) = \binom{t}{t/2}$ for even t and 0 for odd t, so that $\gamma_t = \mathrm{mid}(t)2^{-t}$, and work in $\mathbb{Z}[u][v]$. Two facts. First, $[u^h v^h]\,(u+v)^a(uv+1)^b = \mathrm{mid}(a)\,\mathrm{mid}(b)$ whenever $a+b = 2h$: expanding, $(u+v)^a(uv+1)^b = \sum_{i,j}\binom{a}{i}\binom{b}{j}u^{a-i+j}v^{i+j}$, and $i+j = h = a-i+j$ forces $i = a/2$, $j = b/2$. Second, in any commutative ring, $\sum_{i\le y,\,j\le z}(-1)^i\binom{y}{i}\binom{z}{j}s^{i+j}t^{N-i-j} = (t-s)^y(t+s)^z$ by two binomial theorems. With $s = u+v$, $t = uv+1$ one has $t-s = (u-1)(v-1)$ and $t+s = (u+1)(v+1)$, so applying the first fact termwise to the second,

$$\begin{aligned} 2^N\,\Theta(y,z) &= \sum_{i,j}(-1)^i\binom{y}{i}\binom{z}{j}\,\mathrm{mid}(i+j)\,\mathrm{mid}(N-i-j) \\ &= [u^h v^h]\,(u-1)^y(u+1)^z(v-1)^y(v+1)^z = \Big([u^h](u-1)^y(u+1)^z\Big)^2 \end{aligned}$$

when $N = 2h$, and 0 when N is odd (every product $\mathrm{mid}(i+j)\mathrm{mid}(N-i-j)$ vanishes). For N even and y odd the coefficient vanishes by the last assertion of Lemma 4.4, as does $c_{y,z}$; for $y = 2p$, $z = 2q$ it is $\pm F(p,q)$ and von Szily's identity gives $\Theta(y,z) = 2^{-N}\big(\mathrm{sc}(p,q)\big)^2 = 2^N c_{y,z}$. □

Identity (7) was independently checked by machine for every pair with $y+z \le 20$, and the second proof is machine-checked for all y, z; it reproduces the source's printed weight ($\Theta(2,2) = \frac{1}{4} = 2^4 c_{2,2}$) and fails when $c_{2,0}$ is doubled.

4.3. **The constants of the conjecture.**

Proposition 4.6. *For all $y, z \ge 0$: $c_{y,z} = c_{z,y}$; $c_{y,z} = 0$ as soon as y or z is odd; and*

$$c_{2p,2q} = \left(\frac{\mathrm{sc}(p,q)}{4^{\,p+q}}\right)^2, \qquad \mathrm{sc}(p,q) = \frac{(2p)!(2q)!}{p!\,q!\,(p+q)!}.$$

Proof. The first two are read off (5). For the third, at $y = 2p$, $z = 2q$ the numerator of (5) is $((2p)!(2q)!)^2$ and the denominator is $(p!\,q!\,(p+q)!\,2^{2(p+q)})^2$, and $2^{2(p+q)} = 4^{p+q}$. □

So the constants of Conjecture 6.3 are the squares of normalised super Catalan numbers. Two consequences are worth stating: exchanging the roles of p^- and p^+h leaves the right-hand side of (5) unchanged, and the reason von Szily's identity appears in Lemma 4.5 is that the quantity the proof actually produces, $\mathrm{CT}_t[D]$, is $\pm\sqrt{c_{y,z}}$ up to normalisation — the square in (5) being the square in $\Theta = 2^{-N}(\mathrm{CT}_t[D])^2$, i.e. the two blocks A and B of the stabiliser $G_x = S_A \times S_B$.

5. Conjecture 6.3 holds for every n

This section proves the conjecture in general. It is an ordinary mathematical argument and is not machine-checked; §9 says exactly which of its ingredients are.

Theorem 5.1. *For every $n \ge 0$, with $c_{y,z}$ as in (5),*

$$K_n = \sum_{x+y+z=n} c_{y,z} f(x,y,z).$$

Equivalently, $\widetilde{K}_n[S,T] = c_{|S|,|T\setminus S|}$ for $S \subseteq T$ and 0 otherwise.

Proof. By Proposition 3.2 it suffices to prove (6). Fix $S, T \subseteq [n]$. By Definition 2.1,

$$\widetilde{K}_n[S,T] = 2^{-n} \sum_x (-1)^{|S\cap x|} \sum_y K(x,y)(-1)^{|T\cap y|}. \tag{8}$$

Step 1: the inner sum is a cycle-parity probability. Fix x and let Y be one step of the chain from x. Conditionally on the permutation $\sigma \in G_x$ chosen in the first half-step, Y is constant on each cycle of σ with independent uniform labels, so

$$\mathbb{E}\big[(-1)^{|T\cap Y|} \,\big|\, \sigma\big] = \prod_{C \text{ cycle of } \sigma} \tfrac{1}{2}\big(1 + (-1)^{|T\cap C|}\big) = \mathbf{1}\{\text{every cycle of } \sigma \text{ meets } T \text{ evenly}\}.$$

Hence $\sum_y K(x,y)(-1)^{|T\cap y|}$ is the probability that a uniform $\sigma \in G_x$ has every cycle meeting T in an even number of points. Writing B for the support of x and $A = [n] \setminus B$, one has $G_x = S_A \times S_B$ and the cycles of σ lie inside A or inside B, so the probability factorises and Lemma 4.1 gives

$$\sum_y K(x,y)(-1)^{|T\cap y|} = q(|A|, |T\cap A|)\, q(|B|, |T\cap B|) = \gamma_{|T|-|T\cap B|}\, \gamma_{|T\cap B|}. \tag{9}$$

This is the step where the block sizes disappear.

Step 2: the outer sum is triangular and free of n. Substituting (9) into (8) and summing over x, i.e. over its support $B \subseteq [n]$,

$$\widetilde{K}_n[S,T] = 2^{-n} \sum_{B\subseteq[n]} (-1)^{|S\cap B|} \gamma_{|T|-|T\cap B|}\, \gamma_{|T\cap B|}.$$

Partition $[n]$ into $S\cap T$, $S\setminus T$, $T\setminus S$ and the remainder R, and let B contain i_1, i_2, i_3, i_4 points of the four parts. Then $|T\cap B| = i_1 + i_3$ and $(-1)^{|S\cap B|} = (-1)^{i_1+i_2}$, so the summand depends on i_2 only through the sign, and the i_2-sum is $\sum_{i_2} (-1)^{i_2} \binom{|S\setminus T|}{i_2}$, which vanishes unless $S \setminus T = \emptyset$. Hence $\widetilde{K}_n[S,T] = 0$ unless $S \subseteq T$.

For $S \subseteq T$ put $y = |S|$, $z = |T \setminus S|$ and $d = n - |T|$. The i_4-sum contributes 2^d, and with $i = i_1$, $j = i_3$ what remains is exactly (7):

$$\widetilde{K}_n[S,T] = 2^{-n}\, 2^d \sum_{i \le y} \sum_{j \le z} (-1)^i \binom{y}{i} \binom{z}{j} \gamma_{i+j}\gamma_{y+z-i-j} = 2^{-(y+z)}\, \Theta(y,z).$$

The parameter n has disappeared.

Step 3: evaluation. By Lemma 4.5, $2^{-(y+z)}\Theta(y,z) = c_{y,z}$, which is (6). □

The two steps of the proof that carry n — the two-block factorisation of Step 1 and the four-region regrouping of Step 2 — are bookkeeping; the arithmetic they leave behind is a finite sum in four region sizes, and that sum is machine-checked for all values of its parameters. With s_1, s_2, s_3, s_4 the sizes of $S \cap T$, $S \setminus T$, $T \setminus S$ and $R = [n] \setminus (S \cup T)$, and $i_1, \dots, i_4$ the numbers of points of B in the four parts, substituting (9) into (8) and grouping the subsets B by $(i_1, \dots, i_4)$ gives $\widetilde{K}_n[S,T] = 2^{-n}\, \mathcal{W}(s_1, s_2, s_3, s_4)$ with

$$\mathcal{W}(s_1,s_2,s_3,s_4) = \sum_{i_1 \le s_1} \sum_{i_3 \le s_3} \sum_{i_2 \le s_2} \sum_{i_4 \le s_4} (-1)^{i_1+i_2} \binom{s_1}{i_1}\binom{s_2}{i_2}\binom{s_3}{i_3}\binom{s_4}{i_4} \gamma_{i_1+i_3}\, \gamma_{s_1+s_3-i_1-i_3}. \tag{10}$$

Theorem 5.2. *For all $s_1, s_2, s_3, s_4 \ge 0$,*

$$\mathcal{W}(s_1,s_2,s_3,s_4) = \mathbf{1}\{s_2 = 0\} \cdot 2^{s_4}\, \Theta(s_1,s_3),$$

so that $\mathcal{W}(s_1,s_2,s_3,s_4) = 0$ whenever $s_2 \neq 0$ and $\mathcal{W}(s_1,0,s_3,s_4) = 2^{s_1+s_3+s_4}\, c_{s_1,s_3}$.

Proof. The summand depends on i_2 only through $(-1)^{i_2}\binom{s_2}{i_2}$, whose sum over i_2 is $\mathbf{1}\{s_2 = 0\}$, and on i_4 only through $\binom{s_4}{i_4}$, whose sum is 2^{s_4}; what remains is (7) with $(y,z) = (s_1, s_3)$, and Lemma 4.5 evaluates it. □

With $n = s_1 + s_2 + s_3 + s_4$ this is (6) for every n: the entry vanishes unless $S \subseteq T$, and for $S \subseteq T$ it is $c_{|S|,|T\setminus S|}$, with n absent. Theorem 5.2 is machine-checked as stated, for all four parameters, and so are its ingredients; the identification of $2^n \widetilde{K}_n[S,T]$ with $\mathcal{W}$ — the transform over subsets of $[n]$, the two-block factorisation and the regrouping — is the part of the proof of Theorem 5.1 that is not, together with Lemma 3.1 in its n-fold form and Proposition 3.2. The sum is not degenerate: $\mathcal{W}(0,0,0,0) = 1$, $\mathcal{W}(0,0,0,3) = 8$, $\mathcal{W}(2,0,0,0) = 1$, $\mathcal{W}(2,1,0,0) = 0$ while $\mathcal{W}(2,0,0,0) \neq 0$, and $\mathcal{W}(1,0,1,0) = 0$ because $c_{1,1} = 0$ — the correction of §6 seen inside the normal form.

Theorem 3.3 and Theorem 7.2 below are instances of Theorem 5.1; they were obtained before it, and they remain the machine-checked part of the record.

6. Two consequences

6.1. **The eigenvalue table.** A matrix triangular for a partial order is triangular for any linear extension of it, so its eigenvalues are its diagonal entries with multiplicity. The diagonal of (6) is $\widetilde{K}_n[T,T] = c_{|T|,0}$.

Corollary 6.1. *The spectrum of K_n is $\{c_{|T|,0} : T \subseteq [n]\}$ with multiplicity. Since $c_{2k,0} = \binom{2k}{k}^2 / 2^{4k} = \beta_k$ and $c_{\ell,0} = 0$ for odd ℓ, the eigenvalue β_k occurs $\binom{n}{2k}$ times and, for $n \ge 1$, the eigenvalue 0 occurs $\sum_{\ell \text{ odd}} \binom{n}{\ell} = 2^{n-1}$ times: this is the eigenvalue list (3) of [2].*

The corollary is not new — (3) is Theorem 1.2 of [2] — but the derivation is, and it is a sharp consistency check on (6): an error anywhere in the transform would have to conspire to leave the published multiplicities intact. The counts were also confirmed by direct computation of the diagonal for every $n \le 8$ and every $k \le 4$.

6.2. **A correction to a remark of the source.** The sentence quoted in §1, that the constants $c_{k,k}$ are exactly the eigenvalues β_k, does not hold for the constants (5) defines.

Proposition 6.2. *For every $k \geq 0$ one has $c_{2k,0} = c_{0,2k} = \beta_k$. By contrast $c_{k,k} \neq \beta_k$ for every odd k, since $c_{k,k} = 0 < \beta_k$ there, and also for $k = 2, 4, 6, 8$; among $0 \leq k \leq 8$ the only index with $c_{k,k} = \beta_k$ is $k = 0$. The first discrepancies are $c_{1,1} = 0 \neq \frac{1}{4} = \beta_1$, $c_{2,2} = \frac{1}{64} \neq \frac{9}{64} = \beta_2$ and $c_{3,3} = 0 \neq \frac{25}{256} = \beta_3$.*

Proof. By Proposition 4.6, $c_{2k,0} = (\mathrm{sc}(k,0)/4^k)^2 = (\binom{2k}{k}/4^k)^2 = \binom{2k}{k}^2/2^{4k} = \beta_k$, and $c_{0,2k} = c_{2k,0}$ by symmetry. For odd k, $c_{k,k} = 0$ while $\beta_k > 0$. The even cases $k = 2, 4, 6, 8$, and the tabulated discrepancies, are exact rational computations. □

The weights that are the eigenvalues are therefore $c_{2k,0} = c_{0,2k}$, which is also what Corollary 6.1 predicts, the diagonal entries of the normal form being $c_{|T|,0}$. The remark opens the closing paragraph of a discussion the authors flag as speculative, and no result of [1] depends on it.

7. Direct verification and its reach

Independently of §5, the identity (5) can be checked outright, and we record how far. The literal check is expensive — both sides are $2^n \times 2^n$ rational matrices and the right-hand side is a sum over 3^n words — but both sides are functions of the joint type alone, which collapses the identity to $\binom{n+3}{3}$ scalar equations.

Proposition 7.1. *For a pair (x, y) of joint type (a_0, a_1, b_0, b_1), the entry of the right-hand side of (5) at (x, y) equals*

$$\mathrm{rhs}(a_0, a_1, b_0, b_1) = 2^{-n} \sum_{y', z'} c_{y', z'} \, [u^{y'} v^{z'}](1 + u + v)^{a_0}(1 - u - v)^{a_1}(1 - u + v)^{b_0}(1 + u - v)^{b_1}.$$

Proof. Each word contributes the product over coordinates of the corresponding entry of its letter, and by (4) that entry is $\frac{1}{2}$ for p^+, $\frac{1}{2}(-1)^{x_i + y_i}$ for p^- and $\frac{1}{2}(-1)^{y_i}$ for p^+h. Marking the letters p^- and p^+h by u and v, the sum over words of a coordinate is $1 + u(-1)^{x_i+y_i} + v(-1)^{y_i}$, which is $1 + u + v$, $1 - u - v$, $1 - u + v$, $1 + u - v$ according as (x_i, y_i) is $(0,0), (0,1), (1,0), (1,1)$; collecting the factor 2^{-n} and weighting the bidegree (y', z') by $c_{y',z'}$ gives the formula. □

Theorem 7.2. *For every $n \leq 24$ and every quadruple (a_0, a_1, b_0, b_1) of nonnegative integers with $a_0 + a_1 + b_0 + b_1 = n$,*

$$w(a_0, a_1)\, w(b_0, b_1) = \mathrm{rhs}(a_0, a_1, b_0, b_1).$$

Proof. Exhaustive computation over all $\binom{n+3}{3}$ types for each $n \leq 24$ — 2925 types at $n = 24$ — with the left side evaluated by Proposition 2.2 and the right side by an exact integer dynamic program over the bidegree (y', z'), followed by a single rational division. Together with Proposition 2.2 and Proposition 7.1, this is (5) at every $n \leq 24$; the two bridges to the literal matrices were in addition checked directly, for $n \leq 6$ on the left and $n \leq 5$ on the right. For every n the statement follows from Theorem 5.1. □

The type reduction is what makes the range affordable: at $n = 24$ it replaces 2^{48} matrix entries by 2925 scalar identities. The source reports (5) "checked up to $n = 10$".

Proposition 7.3. *For each n with $1 \leq n \leq 7$ the identity (5) holds, with both sides built literally: the left from the two-step description of the process, the right from the enumeration of all 3^n words.*

These seven cases lie inside the range [1] reports as checked, and we read them as a reproduction of that data rather than as new information; what is new in them is only that they are certified by a proof assistant and that both sides were coded independently. The following controls say that the objects involved are not degenerate and that the identity is not robust to perturbation — so that the verifications above have content.

Proposition 7.4. *Doubling the single weight $c_{2,0}$ breaks (5) at $n = 2$ and at $n = 3$. Deleting the parity case distinction in (5), which makes $c_{1,1} = \frac{1}{16}$, breaks it at $n = 2$; deleting the p^+h terms breaks it at $n = 2$. Replacing the sum over orderings by the multinomial coefficient times the sorted word gives a true identity at $n = 2$ and a false one at $n = 3$. Finally K_n is stochastic for $n \le 5$, and $K_3 \neq K_3^{\mathsf{T}}$, so the identity is a statement about K and not about its transpose.*

The sorted-word control is the sharp one: it locates $n = 3$ as the first place where the sum over orderings of (5) is doing work, every word of length 2 being its own sorted form.

8. The k-ary chain and Conjecture 6.4

8.1. **The chain, and two facts about it.** Fix $k \ge 2$ and let $\mathfrak{X} = (\mathbb{Z}/k)^n$ with S_n permuting coordinates. For $x \in \mathfrak{X}$ and $c \in \mathbb{Z}/k$ put $A_c(x) = \{i : x_i = c\}$, so that $G_x = \prod_c S_{A_c(x)}$, and let $c(\sigma)$ count the cycles of σ. One step from x picks $\sigma \in G_x$ uniformly and labels each cycle of σ uniformly and independently from $\mathbb{Z}/k$, so (1) reads

$$K(x,y) = \frac{1}{|G_x|} \sum_{\sigma \in G_x \cap G_y} k^{-c(\sigma)}, \tag{11}$$

and we write K_n for the $k^n \times k^n$ matrix, acting on functions by $(K_n f)(x) = \sum_y K(x,y) f(y)$. Words $a \in \{0, \dots, k-1\}^n$ index the standard basis, and $\operatorname{supp} a = \{i : a_i \neq 0\}$. Conjecture 6.4 of [1], quoted in §1, is about the multiplicities of the eigenvalues of K_n as n varies with k fixed.

Two facts from [2] are used. The chain is reversible for the distribution $\pi(x) \propto |G_x|$, so K_n is self-adjoint in $\ell^2(\pi)$, hence diagonalisable with real spectrum, and "multiplicity" may be read as algebraic or geometric multiplicity indifferently. (The spectrum is in fact non-negative [3, Corollary 3.9]; this is not used.) And the process lumps to coordinates: [2, Proposition 6.1], stated there for general k with the remark that the binary proof generalises directly, reads

> "The restriction of the Burnside process on (C_k^n, S_n) to any $m \le n$ of its coordinates is also a Markov chain, and its transition probabilities are exactly given by the Burnside process on (C_k^m, S_m)."

In the language of functions: for $U \subseteq [n]$ let W_U be the space of functions of x that depend only on $x|_U$; then $K_n W_U \subseteq W_U$, and under the identification of W_U with the functions on $(\mathbb{Z}/k)^U$ the restriction $K_n|_{W_U}$ is $K_{|U|}$.

8.2. **The closed kernel and the cycle-residue lemma.** The two ingredients of the binary proof that survive for general k are the closed form of the kernel and the cycle-parity lemma, which becomes a cycle-*residue* lemma. Both are routine; they are recorded because they were checked against the objects they replace.

Proposition 8.1. *Put $R_k(m) = \prod_{i=0}^{m-1} \left(\frac{1}{k} + i\right)$, and for states x, y let $N_{cd} = \#\{i : x_i = c,\ y_i = d\}$ and $n_c = \sum_d N_{cd}$. Then*

$$K(x,y) = \prod_{c \in \mathbb{Z}/k} \frac{\prod_{d \in \mathbb{Z}/k} R_k(N_{cd})}{n_c!}.$$

Proof. $\sigma \in G_x \cap G_y$ exactly when σ preserves each of the k^2 cells $\{i : x_i = c, y_i = d\}$, so $G_x \cap G_y = \prod_{c,d} S_{N_{cd}}$ with $c(\sigma)$ additive over the factors, and $\sum_{\sigma \in S_m} u^{c(\sigma)} = u(u+1)\cdots(u+m-1)$ at $u = 1/k$ gives $R_k(m) = \sum_{\sigma \in S_m} k^{-c(\sigma)}$; finally $|G_x| = \prod_c n_c!$. □

For $k = 2$ this is Proposition 2.2. The closed form was checked entry by entry against the matrix built literally from (11) — walking every permutation of $[n]$, counting its cycles and distributing $k^{-c(\sigma)}/|G_x|$ over the pairs of states it fixes — for $k = 2$ and $n \le 4$, $k = 3$ and $n \le 4$, and $k = 4$ and $n \le 3$.

Lemma 8.2 (cycle residue)**.** *Let $\sigma \in S_m$ be uniform and colour $[m]$ by $\mathbb{Z}/k$ with t_r points of colour r for $r = 1, \dots, k-1$ and m_0 points of colour 0; put $t = \sum_r t_r$. The probability that every cycle of σ has*

colour sum $\equiv 0 \pmod k$ *equals*

$$\Gamma_k(t_1,\dots,t_{k-1}) = \frac{t_1!\cdots t_{k-1}!}{t!}\,\big[w_1^{t_1}\cdots w_{k-1}^{t_{k-1}}\big]\,G_k(w),$$

$$G_k(w) = \det \mathrm{Circ}(1,-w_1,\dots,-w_{k-1})^{-1/k} = \prod_{j=0}^{k-1}\Big(1-\sum_{r=1}^{k-1}\omega^{jr}w_r\Big)^{-1/k},$$

$\omega = e^{2\pi i/k}$, *and does not depend on* m_0. *In particular* $\Gamma_2(t) = \gamma_t$, $\Gamma_k(t) = 0$ *unless* $\sum_r rt_r \equiv 0 \pmod k$, *and* $G_2 = (1-w^2)^{-1/2}$, $G_3 = (1-w_1^3-3w_1w_2-w_2^3)^{-1/3}$.

Proof. Multi-sort exponential generating functions, with u_r marking points of colour r and $U = \sum_{r\in\mathbb{Z}/k} u_r$: cycles have generating function $\log\frac{1}{1-U}$, and the condition "colour sum $\equiv 0$" is imposed by the character filter $\mathbf{1}\{s\equiv 0\} = k^{-1}\sum_j \omega^{js}$, i.e. by substituting $u_r \mapsto \omega^{jr}u_r$ and averaging over j. Exponentiating, the permutations all of whose cycles are good have generating function $F_k(u) = \prod_j (1-U_j)^{-1/k}$ with $U_j = \sum_r \omega^{jr}u_r$. Since u_0 occurs with coefficient 1 in every U_j, putting $w_r = u_r/(1-u_0)$ gives $F_k = (1-u_0)^{-1}G_k(w)$, and extracting $[u_0^{m_0}u_1^{t_1}\cdots]$ yields $\binom{m}{t}[w^t]G_k$; dividing by the multinomial coefficient $\binom{m}{m_0,t_1,\dots}$ leaves the stated value, with m_0 gone. The product over j is the circulant determinant $\prod_j\sum_r c_r\omega^{jr}$ for $c = (1,-w_1,\dots,-w_{k-1})$. □

Exactly as in Step 1 of the proof of Theorem 5.1, for a character $\chi_T(y) = \omega^{\langle T,y\rangle}$ one gets

$$\sum_y K(x,y)\chi_T(y) = \prod_{c\in\mathbb{Z}/k}\Gamma_k\big(t_c(x,T)\big), \qquad t_c(x,T)_r = \#\{i : x_i = c,\ T_i = r\},$$

and the independence of m_0 is what removes n. Three finite forms of the lemma were checked by machine, with the determinant computed from its definition as a signed sum over permutations so that no root of unity occurs: a counting recursion for Γ_k against brute-force enumeration of S_m for $k = 2,3,4$ and every type of total weight $t \le 6$; the m_0-independence by enumeration for $k = 2,3$ and every colouring of at most 7 points; and the generating-function formula for $k = 2,3$ to total degree 6 and $k = 4$ to total degree 4, together with the two displayed determinants. Some values: $\Gamma_3(1,1) = \frac12$, $\Gamma_3(3,0) = \Gamma_3(2,2) = \frac13$, $\Gamma_3(3,3) = \frac{23}{90}$.

8.3. **An integer basis, and the block decomposition.** The binary proof continued with the Walsh transform; for general k the character basis of $\mathbb{Z}/k$ would do, but it needs roots of unity, and only one property of it is used: that it is adapted to the supports of words. There is an integer basis with that property. On functions on $\mathbb{Z}/k$ take

$$g_0 = 1 \ \text{ (the constant function)}, \qquad g_r = \delta_r \ \ (r = 1,\dots,k-1),$$

and let G be the $k\times k$ matrix whose a-th column is g_a, so $G[c,a] = 1$ if $a = 0$ and $G[c,a] = \mathbf{1}\{c = a\}$ otherwise; G is unimodular, with $G^{-1}[a,c] = \mathbf{1}\{c = 0\}$ for $a = 0$ and $G^{-1}[a,c] = \mathbf{1}\{c = a\} - \mathbf{1}\{c = 0\}$ for $a \ge 1$. The tensor $g_a = g_{a_1}\otimes\cdots\otimes g_{a_n}$, i.e. the function $x \mapsto \prod_i g_{a_i}(x_i)$, depends on x only through $x|_{\operatorname{supp} a}$, and $\{g_a : \operatorname{supp} a \subseteq U\}$ is a basis of W_U. Put

$$C_n = (G^{-1})^{\otimes n}\,K_n\,G^{\otimes n}, \qquad \text{so that} \qquad K_n g_a = \sum_{a'} C_n[a',a]\,g_{a'},$$

and let D_t be the $(k-1)^t\times(k-1)^t$ submatrix of C_t on the words of full support, $\{1,\dots,k-1\}^t$.

Proposition 8.3 (block decomposition)**.** *For every* $k \ge 2$, *every* $n \ge 0$ *and all words* a, a' *of length* n:

(T) $C_n[a',a] = 0$ *unless* $\operatorname{supp} a' \subseteq \operatorname{supp} a$;
(N) *if* $\operatorname{supp} a' \subseteq \operatorname{supp} a = U$ *and* $|U| = t$, *then* $C_n[a',a] = C_t[a'|_U, a|_U]$.

Consequently C_n, *with its words ordered by* $|\operatorname{supp}|$, *is block triangular, its level-*t *diagonal block is block diagonal with* $\binom{n}{t}$ *copies of* D_t, *and*

$$\operatorname{charpoly} K_n(z) = \prod_{t=0}^{n}\big(\operatorname{charpoly} D_t(z)\big)^{\binom{n}{t}}, \qquad \operatorname{mult}_{K_n}(\lambda) = \sum_{t=0}^{n}\operatorname{mult}_{D_t}(\lambda)\binom{n}{t},$$

with mult *the algebraic multiplicity and D_t independent of n. Moreover $D_t[a',a]$ depends on (a',a) only through the $(k-1)\times(k-1)$ joint type $b_{rd}=\#\{i: a'_i=r,\ a_i=d\}$, and*

(12)

$$D_t[b]=\sum_{0\le e\le b}(-1)^{|e|}\Big(\prod_{r,d}\binom{b_{rd}}{e_{rd}}\Big)K_{\mathrm{type}}\big(N(e)\big),\qquad N(e)_{0,d}=\sum_r e_{rd},\ \ N(e)_{r,d}=b_{rd}-e_{rd},\ \ N(e)_{c,0}=0,$$

where $K_{\mathrm{type}}(N)$ is the right-hand side of Proposition 8.1 as a function of the $k\times k$ type N.

Proof. Let $\operatorname{supp} a\subseteq U$, $|U|=t$, and write $g_a(x)=g_{a|_U}(x|_U)$, which holds because $g_0=1$. Proposition 6.1 of [2] says that from any state x the restriction to U of one step of the chain is distributed as one step of the chain on $(\mathbb{Z}/k)^U$ from $x|_U$ (its proof there erases the coordinates outside U from the cycle notation of σ), i.e. $\sum_{y:\,y|_U=z}K_n(x,y)=K_t(x|_U,z)$ for every $x\in(\mathbb{Z}/k)^n$ and $z\in(\mathbb{Z}/k)^U$; hence for $f\in W_U$ with $f(x)=h(x|_U)$ one has $(K_nf)(x)=\sum_z K_t(x|_U,z)h(z)$, so

$$(K_ng_a)(x)=(K_tg_{a|_U})(x|_U)=\sum_b C_t[b,a|_U]\,g_b(x|_U)=\sum_{\operatorname{supp}a'\subseteq U}C_t[a'|_U,a|_U]\,g_{a'}(x),$$

the sum over $b\in\{0,\dots,k-1\}^U$ being rewritten over the words a' of length n supported in U. Comparing with $K_ng_a=\sum_{a'}C_n[a',a]g_{a'}$ and using that the $g_{a'}$ are a basis gives (T) and (N) with $U=\operatorname{supp}a$. Ordering words by $|\operatorname{supp}|$, (T) makes C_n block triangular; inside the level-t block, $\operatorname{supp}a'\subseteq\operatorname{supp}a$ with $|\operatorname{supp}a'|=|\operatorname{supp}a|$ forces $\operatorname{supp}a'=\operatorname{supp}a$, so the block is block diagonal with one block per t-subset U, and by (N) that block is D_t. Since $G^{\otimes n}$ is invertible, K_n and C_n have the same characteristic polynomial, which is the product of those of the diagonal blocks. For the type formula, the column of D_t at a full-support word a is $(G^{-1})^{\otimes t}$ applied to $x\mapsto K_t(x,a)$ (as $g_a=\delta_a$), read at the full-support rows; a row a' of $(G^{-1})^{\otimes t}$ is $\prod_i(\mathbf{1}\{x_i=a'_i\}-\mathbf{1}\{x_i=0\})$, and expanding the product over the set e of coordinates at which the second term is taken, grouping by the joint type, gives (12). □

Proposition 8.3 is ordinary mathematics, resting on the lumping proposition of [2], and it is not machine-checked; its instances are. (The design record behind this paper reaches the same statement by a different route, through Lemma 8.2 in the character basis of $\mathbb{Z}/k$; the proof above needs neither.)

Theorem 8.4. *The following hold exactly over $\mathbb{Q}$. $G^{-1}G=GG^{-1}=1$ over $\mathbb{Z}$ for $k=2,3,4,5$. Statements (T) and (N) hold at every one of the k^{2n} index pairs of C_n for $k=2$ and $n\le 6$, $k=3$ and $n\le 5$, and $k=4$ and $n\le 4$, with C_n computed from Proposition 8.1 by conjugation one tensor factor at a time, a procedure that agrees with the dense Kronecker products $(G^{-1})^{\otimes n}K_nG^{\otimes n}$ for $k=3$, $n\le 3$ (and, for the right-hand factor, for $k=4$, $n\le 2$) and satisfies $(G^{-1})^{\otimes n}G^{\otimes n}=1$ in both ranges. The block D_t read off C_t equals the type formula (12) for $k=2$ and $t\le 6$, $k=3$ and $t\le 5$, $k=4$ and $t\le 4$. For $k=2$ and $t\le 6$, D_t is the 1×1 matrix $(c_{t,0})$, i.e. $\beta_{t/2}$ for even t and 0 for odd t. Finally $\sum_t\binom{n}{t}(k-1)^t=k^n$ for $k\le 6$, $n\le 6$.*

Proof. Exhaustive computation: the unimodularity and the dimension count are integer identities decided inside the kernel; the rest are identities between arrays of exact rationals. The largest is $k=3$, $n=5$, a 243×243 conjugation and 59049 entry comparisons against the six smaller levels; $k=4$, $n=4$ is 256×256. □

The $k=2$ case says that the blocks of Proposition 8.3 are the diagonal constants of (6), so the binary paper is the case in which every D_t is a scalar; for $k\ge 3$ the n-free object is a matrix of size $(k-1)^t$, and there is no scalar identity — no von Szily, no super Catalan number — to look for.

Corollary 8.5. *For every k and n, the multiplicity of every eigenvalue of K_n is a non-negative integer combination $\sum_t m_t\binom{n}{t}$ with coefficients $m_t=\operatorname{mult}_{D_t}(\lambda)$ independent of n. Conjecture 6.4 holds for a given k if and only if the nonzero spectra of $D_0,D_1,D_2,\dots$ are pairwise disjoint, and then b_λ is the unique level t with $\lambda\in\operatorname{spec}D_t$ and $a_\lambda=\operatorname{mult}_{D_{b_\lambda}}(\lambda)$. In particular, if no two of $D_0,\dots,D_N$ share a nonzero eigenvalue, the conjecture holds for every $n\le N$. For $k=2$ it holds for every n.*

Proof. The first sentence is Proposition 8.3. If the nonzero spectra are disjoint, the sum has one term. Conversely, if $\operatorname{mult}_{K_n}(\lambda)=a\binom{n}{b}$ for all n, then $\sum_t m_t\binom{n}{t}=a\binom{n}{b}$ for all n, and the functions $n\mapsto\binom{n}{t}$ are

TABLE 1. The nonzero irreducible factors of charpoly D_t over $\mathbb{Q}$ for $k = 3$, $t \le 8$, with their isotypic labels. $D_1 = 0$.

t	factors $(\mu,\ f^\mu)$
0	$z-1$ ((0), 1)
2	$3z-1$ ((2), 1)
3	$27z-2$ ((3), 1)
4	$27z-5$ ((4), 1); $18z-1$ ((2,2), 2)
5	$81z-4$ ((5), 1); $27z-1$ ((4,1), 4)
6	$59049z^2-8613z+160$ ((6), 1); $729z-23$ ((4,2), 9)
7	$2187z-80$ ((7), 1); $1215z-32$ ((6,1), 6); $3645z-76$ ((5,2), 14)
8	$1594323z^2-172773z+2464$ ((8), 1); $2187z-32$ ((7,1), 7); $10935z-233$ ((6,2), 20); $243z-4$ ((4,4), 14)

linearly independent on $\mathbb{Z}_{\ge 0}$, so $m_t = a\,\mathbf{1}\{t = b\}$. For $k = 2$ the D_t are the scalars $c_{t,0}$ of Theorem 8.4 and Corollary 6.1, whose nonzero values β_j are pairwise distinct. □

Only levels $t \le n$ contribute at n, which is why finite certificates below give the conjecture in a range of n; and $D_1 = 0$ (Proposition 8.11), which is why b_λ is never 1.

8.4. $k = 3$**: the levels are pairwise coprime up to** $t = 16$**.** D_t commutes with the permutations of the t coordinates, so by Schur–Weyl duality for $GL_{k-1} \times S_t$ on $(\mathbb{Q}^{k-1})^{\otimes t}$ it splits as $\bigoplus_\mu \mathrm{id}_{S^\mu} \otimes D_t^\mu$ over the partitions $\mu \vdash t$ with at most $k-1$ rows, S^μ the Specht module of dimension f^μ, and $\operatorname{spec} D_t = \biguplus_\mu f^\mu \operatorname{spec} D_t^\mu$. For $k = 3$ the blocks $D_t^{(t-m,m)}$ have size $t - 2m + 1$, and their characteristic polynomials were factored exactly over $\mathbb{Q}$ for every $t \le 16$ outside the proof assistant. The nonzero factors are 62 primitive integer polynomials $P_1, \dots, P_{62}$ of degree at most 3 (38 linear, 19 quadratic, 5 cubic), each recorded with its level t, its partition μ and $f^\mu = \binom{t}{m} - \binom{t}{m-1}$; Table 1 lists the levels $t \le 8$. What is proved inside the proof assistant is the following.

Theorem 8.6. *The* 62 *recorded polynomials are pairwise distinct, and for every one of the* 1891 *pairs* $i < j$ *there are recorded* $u, v \in \mathbb{Z}[z]$ *and* $c \in \mathbb{Z} \setminus \{0\}$ *with*

$$u\,P_i + v\,P_j = c \qquad \text{in } \mathbb{Z}[z];$$

the list of certificates covers exactly the pairs $i < j$*, once each. Hence no two of the* 62 *polynomials have a common root. Every part of this is integer polynomial arithmetic and is decided inside the kernel, with no compiled evaluation.*

Proof. A common root λ of P_i and P_j would give $c = u(\lambda)P_i(\lambda) + v(\lambda)P_j(\lambda) = 0$. The identities are checked by multiplying out; the certificates total 70 350 decimal digits, the largest integer having 49 digits, which is small because the isotypic refinement keeps every degree at most 3. How the certificates were found is immaterial. □

Granting the cited factorisation, no two levels $t \ne t' \le 16$ share a nonzero eigenvalue, so by Corollary 8.5: *Conjecture 6.4 holds for* $k = 3$ *and every* $n \le 16$, with b_λ the unique level carrying λ and $a_\lambda = \sum_\mu f^\mu \operatorname{mult}_{D^\mu_{b_\lambda}}(\lambda)$, a sum of numbers of standard Young tableaux, which is the shape [1] anticipates for a_λ. The source's own computations, in the sentence quoted in §1, reach $n \le 8$ for the orbit chain and $n \le 7$ for the multiplicity of $\frac{1}{18}$; a brute-force modular-rank computation at $n = 8$ was priced at about 27 hours before this reduction. The first irrational eigenvalues appear at $t = 6$, in the symmetric block $59049z^2 - 8613z + 160$, consistent with the remark of [2] that the orbit (Bose–Einstein) chain has irrational eigenvalues already for $k = 3$, $n = 6$.

The identification of the 62 polynomials with the factors of charpoly D_t is the one step of this argument done outside the proof assistant. It can be removed wherever the full block D_t fits, by a single exact matrix identity. Let $R_t = \prod_\mu P_t^\mu$ be the product of the distinct level-t polynomials of the list ($R_1 = 1$); for instance $R_4 = (27z-5)(18z-1) = 486z^2 - 117z + 5$ and $R_8 = (1594323z^2 - 172773z + 2464)(2187z - 32)(10935z - 233)(243z - 4)$.

Theorem 8.7. *For $k = 3$ and every $t \le 8$, $R_t(D_t)\, D_t = 0$ exactly over $\mathbb{Q}$ on the full $2^t \times 2^t$ matrix D_t.*

Proof. Exact rational arithmetic: D_t is built by the type formula (12) (checked against the block of C_t for $t \le 5$), $R_t(D_t)$ is evaluated by Horner's rule, and every entry of the product is compared with 0. At $t = 8$, a 256×256 block and R_8 of degree 5, the identity is checked one column at a time, $R_8(D_8)(D_8 e_j)$ by Horner on matrix-vector products; the column-wise and the matrix forms agree at every $t \le 6$. □

If $D_t v = \lambda v$ with $v \neq 0$ then $R_t(\lambda)\lambda = 0$, so $\operatorname{spec} D_t \subseteq \{0\} \cup \{\text{roots of } R_t\}$ for $t \le 8$ with no representation theory, and with Theorem 8.6 no two levels $t \neq t' \le 8$ share a nonzero eigenvalue. Hence *Conjecture 6.4 holds for $k = 3$ and every $n \le 8$* with Proposition 8.3 as the only ingredient outside the proof assistant, and for $n \le 5$ not even that (Theorem 8.4). The minimal polynomial of D_t divides zR_t, whose roots are simple when the P_t^μ are irreducible; so, for $t \le 8$, D_t is diagonalisable — consistent with the diagonalisability of K_n, and independent of [3]. The annihilators are needed in full: at $t = 4$ neither $27z - 5$ nor $18z - 1$ annihilates D_4 alone, R_4 does not annihilate D_5, R_7 does not annihilate D_8, and neither does the single factor $243z - 4$ (Proposition 8.11). At $t = 9$ the block is 512×512 and the check is out of reach of the budget used here, which is why $t \le 8$ is Schur–Weyl-free and $9 \le t \le 16$ is not.

8.5. The eigenvalue $\frac{1}{18}$, and the rectangular family.

Theorem 8.8. *Let $k = 3$ and index the rows of the 16×16 block D_4 by the words of $\{1, 2\}^4$ in increasing order of the integer $\sum_i a_i 3^{i-1}$. Then*

$$v_1 = (0,0,0,0,0,1,-1,0,0,-1,1,0,0,0,0,0), \qquad v_2 = (0,0,0,1,0,0,-1,0,0,-1,0,0,1,0,0,0)$$

satisfy $D_4 v = \frac{1}{18} v$, and the 2×2 minor of (v_1, v_2) on the rows in positions 3 and 5 (counting from 0) is -1, so $\frac{1}{18}$ is an eigenvalue of D_4 of geometric multiplicity at least 2. Moreover $R_4(\frac{1}{18}) = 0$ and $R_t(\frac{1}{18}) \neq 0$ for every $t \le 8$ with $t \neq 4$; $R_t(\frac{4}{243}) \neq 0$ for every $t \le 7$ and $R_8(\frac{4}{243}) = 0$; and $R_t(\frac{6857}{885735}) \neq 0$ for every $t \le 11$ while $R_{12}(\frac{6857}{885735}) = 0$.

Proof. Exact rational computation of the two matrix-vector products, the minor and the polynomial values. □

With Theorem 8.7, $\frac{1}{18}$ is an eigenvalue of no D_t with $t \le 8$, $t \neq 4$, so for $n \le 8$ its multiplicity in K_n is $\operatorname{mult}_{D_4}(\frac{1}{18})\binom{n}{4}$: the level $b_{1/18} = 4$ is forced, where [2] could only write "suggesting". The multiplicity in D_4 is at least 2 by the eigenvectors and exactly $2 = f^{(2,2)}$ by the factorisation of Table 1, giving $2\binom{n}{4} = 2, 10, 30, 70$ at $n = 4, 5, 6, 7$ — the source's printed table, derived. Likewise $\frac{4}{243}$ is an eigenvalue of no level $t \le 7$; by the cited factorisation it is the $\mu = (4, 4)$ eigenvalue of D_8 with multiplicity 14, so $\operatorname{mult}_{K_n}(\frac{4}{243}) = 14\binom{n}{8} = 14, 126, 630, 2310$ at $n = 8, \dots, 11$; and $\frac{6857}{885735}$, absent from every level $t \le 11$, is the $\mu = (6, 6)$ eigenvalue of D_{12} with multiplicity 132, so $\operatorname{mult}_{K_n} = 132\binom{n}{12} = 132, 1716, 12012, 60060$ at $n = 12, \dots, 15$ (for $t \le 16$). The three arithmetic tables are checked inside the kernel; the statements about levels $9 \le t \le 12$ rest on the cited factorisation, as in Theorem 8.6.

These three eigenvalues are members of one family. For the rectangular partition $\mu = (a^{k-1})$ at $t = a(k-1)$ the GL_{k-1}-block L^μ is one-dimensional, spanned by $v_a = \bigotimes_{j=1}^{a} \big(\sum_{\pi \in S_{k-1}} \operatorname{sgn}(\pi)\, e_{\pi(1)} \otimes \cdots \otimes e_{\pi(k-1)}\big)$, so D_t acts on $S^\mu \otimes L^\mu$ as a scalar θ_a of multiplicity $f^{(a^{k-1})}$, the number of standard Young tableaux of the $(k-1) \times a$ rectangle. Reading the coefficient of $D_t v_a$ at the word $(1, 2, \dots, k-1)^a$, whose coefficient in v_a is 1, and grouping the a-tuples of permutations by how many times c_π each π occurs, the scalar is computable from the type formula alone:

$$\theta_a = \sum_{\substack{c = (c_\pi)_{\pi \in S_{k-1}} \\ \sum_\pi c_\pi = a}} \Big(\prod_\pi \operatorname{sgn}(\pi)^{c_\pi} \Big) \frac{a!}{\prod_\pi c_\pi!}\, D_{a(k-1)} \Big[\sum_\pi c_\pi P_\pi \Big], \tag{13}$$

P_π the permutation matrix of π read as a joint type; for $k = 3$, $\theta_a = \sum_{u=0}^{a} (-1)^u \binom{a}{u} D_{2a} \left[\begin{smallmatrix} a-u & u \\ u & a-u \end{smallmatrix}\right]$.

Theorem 8.9. *With θ_a defined by* (13) *and D_t by the type formula: for $k=3$, $\theta_1=\theta_3=\theta_5=\theta_7=0$ and*

$$\theta_2=\tfrac{1}{18},\qquad \theta_4=\tfrac{4}{243},\qquad \theta_6=\tfrac{6857}{885735},\qquad \theta_8=\tfrac{357161}{79716150};$$

for $k=4$, $\theta_2=\frac{3}{256}$; for $k=5$, $\theta_2=\frac{3}{1250}$. For $k=3$ the value $\theta_2=\frac{1}{18}$ is the eigenvalue of D_4 exhibited in Theorem 8.8, and $\Gamma_3(2,2)\,\Gamma_3(4,4)^2=\frac{361}{24300}\neq\frac{4}{243}=\theta_4$.

Proof. Exact rational evaluation of (13); the type formula was checked against the block of C_t in the ranges of Theorem 8.4. □

The interpretation of θ_a as an eigenvalue of multiplicity $f^{(a^{k-1})}\binom{n}{a(k-1)}$ in K_n is the Schur–Weyl statement above and stays an ordinary argument; for $k=3$, $f^{(a,a)}=\mathrm{Cat}(a)$, so the multiplicities are $2\binom{n}{4}$, $14\binom{n}{8}$, $132\binom{n}{12}$, $1430\binom{n}{16}$ for $a=2,4,6,8$, and the hook-length formula gives $f^{(2,2,2)}=5$ and $f^{(2,2,2,2)}=14$, hence $5\binom{n}{6}$ for $k=4$ and $14\binom{n}{8}$ for $k=5$. Three independent confirmations that the rectangular scalars are genuine eigenvalues at the predicted level: $18z-1$, $243z-4$ and $885735z-6857$ are the $\mu=(2,2),(4,4),(6,6)$ factors of levels $4,8,12$ in the list of Theorem 8.6, and $256z-3$ is a factor of the minimal polynomial of D_6 for $k=4$ (Theorem 8.10), a 729×729 block built from the process with no representation theory in it. The source prints exactly one value of the family, $\frac{1}{18}$, with its four multiplicities. The vanishing at odd a is observed for $a\le7$ and not proved; no closed form for θ_a is known, and the last assertion of Theorem 8.9 rules out the obvious product of cycle-residue constants, which is why the von Szily endgame of §5 has no analogue here.

8.6. $k=4$**.** Neither source computes anything for $k=4$. For $t\le6$ the minimal polynomial of the $3^t\times3^t$ block D_t was computed exactly over $\mathbb{Q}$, by a Krylov dependence on the exact rational matrix, outside the proof assistant; its nonzero part factors into 11 primitive integer polynomials:

$t=0:$	$z-1$	$t=4:$	$32z-3,\ \ 2048z^2-468z+9$
$t=1:$	none $(D_1=0)$	$t=5:$	$16z-1,\ \ 64z-1,\ \ 256z-21$
$t=2:$	$8z-3$	$t=6:$	$256z-3,\ \ 196608z^2-12416z+107,$
$t=3:$	$8z-1$		$4194304z^3-790528z^2+32316z-315.$

Theorem 8.10. *For $k=4$: every one of the* 55 *pairs among the* 11 *polynomials carries a recorded Bézout identity $uP_i+vP_j=c\in\mathbb{Z}\setminus\{0\}$, decided inside the kernel, so no two of them share a root; and with R_t the product of the level-t polynomials, $R_t(D_t)\,D_t=0$ exactly over $\mathbb{Q}$ on the full $3^t\times3^t$ block for every $t\le5$.*

Hence no two levels $t\neq t'\le5$ share a nonzero eigenvalue, and by Corollary 8.5 *Conjecture 6.4 holds for $k=4$ and every $n\le5$* (Theorem 8.4 for $n\le4$, Proposition 8.3 at $n=5$). This is the first exact statement about the conjecture for any $k\ge4$; the design record behind this paper had floating-point spectra only for $k=4$. At $t=6$ the block is 729×729 and the Horner evaluation of R_6, of degree 6, costs about 2.7×10^9 exact rational operations, over the budget used here; there the identity $R_6(D_6)D_6=0$ was verified modulo the two primes 999983 and 999979 outside the proof assistant, which extends the *computed* range to $n\le6$ without extending the machine-checked one. Two remarks: $256z-3$ at $t=6$ is $\theta_2=\frac{3}{256}$ of Theorem 8.9, found independently in a block that knows nothing of rectangles, with multiplicity $5\binom{n}{6}=5,35,140,420$ at $n=6,\dots,9$; and the irrationalities arrive earlier than for $k=3$ — the first quadratic factor at $t=4$ and the first cubic at $t=6$, against $t=6$ and $t=12$.

8.7. **Controls.**

Proposition 8.11. *(i) Shifting the diagonal entry of the level-t table at the first full-support word by* 1 *breaks the statement (N) of Theorem 8.4 at $n=3$ for $k=3$ and every $t\le3$, and at $n=3$, $t=2$ for $k=4$. (ii) For $k=3$, $n=2$ there are nonzero entries $C_2[a',a]$ with $\operatorname{supp}a'\subsetneq\operatorname{supp}a$, and zero entries with $\operatorname{supp}a\subsetneq\operatorname{supp}a'$: the triangularity is strict and one-sided. (iii) $D_0=(1)$, $D_1=0$, and $D_t\neq0$ for $t=2,3,4$ ($k=3$) and $t=2$ ($k=4$). (iv) For $k=3$ and $n\le3$, K_n is stochastic and $|G_x|K(x,y)=|G_y|K(y,x)$, but K_2 is not symmetric: $K((0,0),(0,1))=\frac{1}{18}$ while $K((0,1),(0,0))=\frac{1}{9}$. (v) $\Gamma_3(t)=0$ for every type of total weight at most 6 with $t_1+2t_2\not\equiv0\pmod3$, and Γ_3 is nonzero on the residue-zero types listed after Lemma 8.2. (vi) The Bézout check is not*

vacuous: $27(18z-1)-18(27z-5)=63$ *is accepted, while changing* 63 *to* 64 *or* 0*, or a coefficient of* u*, is rejected, and the same certificate fails on the pair* $(18z-1, 18z-1)$*; for the non-coprime pair* $18z-1$ *and* $(18z-1)(27z-5)$ *the combination the Euclidean algorithm produces is* 0*. (vii) For* $k=3$*: neither* $27z-5$ *nor* $18z-1$ *annihilates* D_4*,* R_4 *does not annihilate* D_5*,* R_7 *does not annihilate* D_8*, and neither does* $243z-4$*. For* $k=4$*: neither factor of* R_4 *annihilates* D_4 *alone, and* R_4 *does not annihilate* D_5*.*

Control (i) records a trap: the naive perturbation of the $(0,0)$ entry of a level table is never consulted for $t \geq 1$, because a column of full support has no zero digit, so the entry actually shifted is the first full-support diagonal entry.

8.8. **What the data say about a proof.** Corollary 8.5 makes the conjecture cheap to test and, so far, expensive to prove. Two mechanisms that would force level-disjointness were tested against the exact spectra above and both fail. The ranges of the nonzero spectra of successive even levels interleave: for $k=3$, by Table 1, the nonzero eigenvalues of D_4 are $\frac{1}{18}$ and $\frac{5}{27}$ and those of D_6 are $\frac{23}{729}$ and the two roots $0.0218\ldots$ and $0.1240\ldots$ of $59049z^2-8613z+160$, so level 4 lies in $[\frac{1}{18}, \frac{5}{27}] \approx [0.056, 0.185]$ and level 6 in $[0.021, 0.125]$. And the 3-adic valuation of an eigenvalue is not a function of its level: $\frac{2}{27}$, $\frac{5}{27}$ and $\frac{1}{27}$, at levels 3, 4 and 5, all have valuation -3. The eigenvalues are algebraic numbers of growing degree, and the conjecture for $k \geq 3$ is a no-coincidence statement about infinitely many of them. This is an observation from the design record behind this paper, not a theorem, and it is why the reduction, the certified ranges and the rectangular family, rather than a proof, are what is offered here.

9. Verification

Every finite assertion of this paper, and every general assertion marked as such, is checked in the form in which it is stated inside the Lean 4 proof assistant [8]; the development is seventeen modules, in three groups, and contains no `sorry` and declares no axiom of its own. Repository reference to be added at submission.

The binary chain, finite statements (six modules, 978 lines, no external library: rationals, arrays and lists from Lean's core, factorial and binomial coefficient defined locally). Theorem 3.3 ($n \leq 10$), Theorem 7.2 ($n \leq 24$), Proposition 3.4, Proposition 7.3 ($n \leq 7$), Proposition 7.4, and the finite instances quoted beside the general statements: the closed form of Proposition 2.2 ($m \leq 7$ for R, $n \leq 6$ against the literal process), the cycle-parity lemma in permutation form ($m \leq 8$, all $t \leq m$), the core identity ($y+z \leq 20$), von Szily's identity ($p, q \leq 10$), the eigenvalue counts of Corollary 6.1 ($n \leq 8$, $k \leq 4$) and the comparisons of Proposition 6.2 ($k \leq 8$); Proposition 4.6 and the matrix identities of Lemma 3.1 for all arguments. The integral statements among these — the 2×2 algebra, the cleared-denominator von Szily instances, the super Catalan form, the printed weights and the cross-multiplied comparisons — are decided or proved inside the kernel with no compiled evaluation, several with no axioms at all and the others with only `propext` and `Quot.sound`; every rational-valued check in this group, except the symmetry and odd-vanishing halves of Proposition 4.6, which are ordinary proofs, is run by compiled evaluation and carries one compiled-evaluation axiom per check, because rational arithmetic does not reduce inside the kernel.

The binary chain, all parameters (four modules, 1 413 lines, importing the polynomial, power-series and binomial-coefficient parts of Mathlib). Lemma 4.4 in its cleared-denominator form for all p, q, with the integrality of $\mathrm{sc}(p,q)$ and the vanishing of the middle coefficient, and a bridge identifying the finite von Szily statement of the first group as its instance; Lemma 4.5, second proof, for all y, z, stated on the same rational-valued sum and weight as the finite check; Lemma 4.3 for all $t \leq m$, in the closed form with $R(m) = (2m)!/(4^m m!)$; Theorem 5.2 for all $s_1, \ldots, s_4$; and their controls. Every statement in this group depends on at most the axioms `propext`, `Classical.choice` and `Quot.sound` — the general statements on exactly those three, the small value tables and the control that the sign of von Szily's identity is needed on no axiom at all — with no compiled evaluation anywhere: nothing is evaluated, the proofs are symbolic. Not formalised, and stated above as ordinary mathematics: the permutation form of the cycle-parity lemma for general m, the identification $R(m) = \sum_\sigma 2^{-c(\sigma)}$ for general m, the Walsh transform over subsets of $[n]$ with the two-block factorisation and the four-region regrouping, Lemma 3.1 in its n-fold form, Propositions 3.2 and 7.1, Corollary 6.1 in general, and therefore Theorem 5.1 itself.

The k-ary chain (seven modules, 3 592 lines, of which about 2 000 are the recorded factors and certificates; no external library). Proposition 8.1 in the ranges stated after it; Lemma 8.2 in the three finite forms stated after it; Theorem 8.4; Theorem 8.6; Theorem 8.7, with the agreement of the type formula and the block and of the column-wise and matrix forms of the check; Theorem 8.8, with the three multiplicity tables; Theorem 8.9; Theorem 8.10; Proposition 8.11. The statements about integers — the unimodularity of G, the dimension count, the 1891 and 55 Bézout identities with their coverage and distinctness checks, the three multiplicity tables, the list of $k = 4$ levels and the certificate controls — are decided inside the kernel: the coverage, distinctness, unimodularity, multiplicity-table and certificate-control statements depend on no axiom at all, and the two coprimality theorems and the $k = 4$ level list on `propext` alone. Every rational-valued check in this group runs by compiled evaluation and depends on `propext`, `Classical.choice`, `Quot.sound` and its own compiled-evaluation axioms, and on nothing else. Not formalised: Proposition 8.3 and Corollary 8.5 for general n; the Schur–Weyl decomposition of D_t; the exact factorisation identifying the 62 polynomials with the factors of charpoly D_t for $9 \le t \le 16$ and the 11 with the minimal polynomials of the $k = 4$ blocks; the interpretation of θ_a as an eigenvalue of multiplicity $f^{(a^{k-1})}$; and the $t = 6$, $k = 4$ annihilator, verified modulo two primes only.

Remark 9.1 (cost; a measurement, outside the formal development). The six binary modules re-check in about 27 minutes of wall clock with a peak resident set of 1.6 gigabytes, the literal right-hand side of (5) costing a factor of about 12 per step in n (16 seconds at $n = 6$, 257 at $n = 7$, an estimated hour at $n = 8$), which is why Proposition 7.3 stops at $n = 7$. The four all-parameter modules check in under a minute in total, at 2.6 gigabytes. The seven k-ary modules check in about 30 minutes at a peak of 165 megabytes, the two slowest files being the $k = 4$ annihilator at $t = 5$ (a 243×243 block, about 15 minutes) and the $k = 3$ annihilator at $t = 8$ (about 8 minutes); the exact computations outside the proof assistant that produced the factors, certificates, minimal polynomials and eigenvectors took about 13 minutes in total. All figures were measured on a loaded machine and are upper bounds, varying by some 40% between runs.

References

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[2] P. Diaconis, A. Lin and A. Ram, *A curiously slowly mixing Markov chain*, arXiv:2511.01245 [math.PR], v1 3 November 2025, v2 29 December 2025 (not withdrawn, no journal reference; live listing checked 4 September 2026). The conjecture discussed here stood in v1 and is absent from v2, which moved the Schur–Weyl material into [1]; that is why the live home of the conjecture is the math.RT paper. This is also the source of the eigenvalue list (3): its Theorem 1.2, with that number in both v1 and v2. Its Section 6 ("Related chains"), read here from the compiled v2 e-print, carries the lumping statement quoted in §8 as Proposition 6.1, the k-ary multiplicity conjecture as Conjecture 6.2, and the remark on the eigenvalue $\frac{1}{18}$ in the paragraph that follows it.

[3] I. Z. Feng, *The dual Burnside process*, J. Theor. Probab. **39** (2026), no. 4, article 78, doi:10.1007/s10959-026-01539-9 (Crossref: published online 27 August 2026); arXiv:2510.25202 [math.PR], v1 29 October 2025, v4 30 August 2026, whose comment says the text was updated to agree with the published article and whose listing carries the same journal reference (live listing checked 4 September 2026). Read here from the v4 e-print as it compiles: its Corollary 3.9 (p. 11), "Nonnegative spectrum, gaps, and relaxation times", states "All eigenvalues of K and Q are real and nonnegative", K the classical Burnside kernel and Q its dual. The numbering is that of v4; in v1 the corresponding statement is Corollary 3.10, "Gaps and relaxation time agree", and does not assert non-negativity.

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