THE h_4 INEQUALITY BEHIND HUNTER'S CONJECTURE: AN EXACT REAL RELAXATION, TWO ERRATA, AND THE REAL-EXPONENT CONJECTURE OF BRAZITIKOS AND PANDIS

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ABSTRACT. Brazitikos and Pandis (arXiv:2512.12254) prove Hunter's conjecture of 1977: for even n the minimum of the complete homogeneous symmetric polynomial h_{2k} on the unit sphere of $\mathbb{R}^n$ is attained at the half-plus/half-minus vector. The base case $k = 2$ of their induction reduces, through a Lagrange-multiplier argument, to the inequality $F(d) \geq 0$ for $F(d_1, d_2, d_3) = \sum_{\text{cyc}}(d_i + 2)^2(d_i + d_j - d_k)(d_i + d_k - d_j)$, where $d_i + 1$ is the multiplicity of a coordinate value of an extremum, so that $d \in \mathbb{N}^3$. We show that this is a statement about lattice points — it fails on the real orthant — and we determine exactly where it fails: for real $d_1, d_2, d_3 \geq 0$ one has $F(d) \geq 0$ as soon as $d_1 + d_2 + d_3 \geq 2\sqrt{2} - 2$, and the constant is sharp, since $F(t, 0, 0) = t^2(t^2 + 4t - 4) < 0$ for $0 < t < 2\sqrt{2} - 2$. The proof consists of two polynomial identities. A by-product is a uniform proof of the lattice inequality itself, replacing the three-way case analysis of the source, in which two of the three displayed rewritings are wrong: one printed bracket equals -15 at $(5, 4, 0)$ and -2117 at $(20, 19, 2)$ where it is asserted to be non-negative, and the $d_3 = 1$ display differs from $F(d_1, d_2, 1)$ by $(d_1 - d_2)^2(d_1 + d_2) - 1$. We give the repaired forms, both exact identities, and record that a third identity in the same proof is correct only under the convention that an unqualified sum runs over the distinct terms obtained by permuting the indices. We also prove the step $a^2 + b^2 \geq 2/n$ that the source asserts without argument, in a weighted real form, and we certify the algebra behind the source's partial treatment of the case $n = 2$ of its Conjecture 4.8, the real-exponent moment inequality for weighted sums of exponential random variables, noting that its sign-change count requires the hypothesis $q > 2$. All of this is machine-checked in Lean 4 over Mathlib. The conjecture is known for exponents in $[0, 4]$ by work of Brazitikos, Tang and Tkocz; in closing remarks, outside the formal development, we summarise what is known for $q > 4$ and describe planned work on the residual window.

1. INTRODUCTION

Hunter's conjecture. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and an integer $k \geq 0$ let

$$h_k(a) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} a_{i_1} \cdots a_{i_k}$$

be the complete homogeneous symmetric polynomial of degree k. If $X_1, \ldots, X_n$ are independent standard exponential random variables then $k!\, h_k(a) = \mathbb{E}\big(\sum_i a_i X_i\big)^k$ (equation (1.2) of [1]), so questions about h_k on the unit sphere are questions about moments of weighted sums of exponentials. Hunter [3] proved that $h_{2k}(a) \geq \frac{1}{k!\,2^k}\big(\sum_i a_i^2\big)^k$ for every $k \geq 1$ (equation (1.1) of [1]) and conjectured, in the words of [1], that "when n is even, under the normalization $\sum a_i^2 = 1$, the global minimum of h_{2k} is attained at the 'half-plus/half-minus' vector"

$$\tilde{a} = \Big(\underbrace{\tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}}}_{n/2}, \underbrace{-\tfrac{1}{\sqrt{n}}, \ldots, -\tfrac{1}{\sqrt{n}}}_{n/2}\Big).$$

We quote the conjecture in the formulation of [1]; Hunter's paper is cited here through its published record and abstract, and its text was not consulted for the statement. Brazitikos and Pandis [1] prove the conjecture. Their Theorem 3.2 reads:

> *Let n be an even integer, and let $a_1, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i^2 = 1$. Then, for every integer $r \geq 1$,*
>
> $$h_{2r}(a_1, \ldots, a_n) \;\geq\; \frac{\big(\frac{n}{2} + r - 1\big)!}{r! \cdot \big(\frac{n}{2} - 1\big)! \cdot n^r},$$

and this inequality is sharp, with equality achieved if and only if $(a_1,\dots,a_n)$ *is a permutation of the half-plus/half-minus vector* $\tilde a$.

The base case and the polynomial F. The proof of Theorem 3.2 of [1] is an induction on r whose base case $r=2$ is their Proposition 3.1:

> *Let $a_1,\dots,a_n$ be real numbers such that $\sum_{i=1}^n a_i^2=1$. Then h_4 attains its maximum at the vector $\overline{a}=\big(\frac{1}{\sqrt n},\dots,\frac{1}{\sqrt n}\big)$, while it attains its minimum at the "half-plus/half-minus" vector $\tilde a=\big(\underbrace{\tfrac{1}{\sqrt n},\dots,\tfrac{1}{\sqrt n}}_{n/2},\underbrace{-\tfrac{1}{\sqrt n},\dots,-\tfrac{1}{\sqrt n}}_{n/2}\big)$ when n is even, and at a vector of the form $\big(\underbrace{a,\dots,a}_{\frac{n-1}{2}},\underbrace{b,\dots,b}_{\frac{n+1}{2}}\big)$, when n is odd. Here a appears $\frac{n-1}{2}$ times, b appears $\frac{n+1}{2}$ times, and $\frac{a}{b}$ minimizes the function $\dfrac{x^2+1+\big(\frac{n+1}{2}x+\frac{n+3}{2}\big)^2}{\frac{n-1}{2}x^2+\frac{n+1}{2}}$.*

For the minimum, the proof in [1] runs as follows. By a Lagrange-multiplier argument (their Proposition 3.3) an extremum of h_4 on the sphere takes at most three distinct coordinate values a,b,c, with multiplicities $\gamma_1,\gamma_2,\gamma_3\ge 1$, $\gamma_1+\gamma_2+\gamma_3=n$; a direct computation gives $h_4(\tilde a)=\frac18+\frac{1}{4n}$, and the claim reduces to $a^2+b^2+c^2\ge\frac{2}{n-1}$ under the two constraints $(\gamma_1+1)a+(\gamma_2+1)b+(\gamma_3+1)c=0$ and $\gamma_1a^2+\gamma_2b^2+\gamma_3c^2=1$. Writing $x=a/c$, this becomes the non-negativity of a quadratic polynomial in x, i.e. a non-negative leading coefficient and a non-positive discriminant. With $d_i:=\gamma_i-1\ge 0$ the leading coefficient is handled by the displayed identity

$$(d_2+2)^2(d_2+d_3-d_1)+(d_1+2)^2(d_1+d_3-d_2)=(d_1-d_2)^2(d_1+d_2+4)+d_3\big[(d_2+2)^2+(d_1+2)^2\big], \tag{1}$$

and for the discriminant the source says: "Due to symmetry, we may assume that $\gamma_1\ge\gamma_2\ge\gamma_3$. Then substituting $n=\gamma_1+\gamma_2+\gamma_3$, it suffices to prove that

$$\sum(d_1+2)^2(d_1+d_2-d_3)(d_1+d_3-d_2)\ \ge\ 0."$$

Writing the sum out, this is the inequality

$$F(d_1,d_2,d_3):=\sum_{\text{cyc}}(d_i+2)^2(d_i+d_j-d_k)(d_i+d_k-d_j)\ \ge\ 0\qquad\text{for }(d_1,d_2,d_3)\in\mathbb N^3, \tag{2}$$

which [1] establishes by a case analysis on $d_3=0$, $d_3=1$ and $d_1,d_2,d_3\ge 2$. We do not re-examine the reduction from h_4 to (1) and (2); everything below concerns these two algebraic facts themselves.

What this paper does. Our main result is that (2) is genuinely a statement about lattice points, and we locate its real failure set exactly.

Theorem (Theorem 3.1 below)**.** *For real $d_1,d_2,d_3\ge 0$ with $s=d_1+d_2+d_3$: if $s\ge 2\sqrt2-2$ then $F(d_1,d_2,d_3)\ge 0$. The constant cannot be lowered: $F(t,0,0)=t^2(t^2+4t-4)<0$ for every $0<t<2\sqrt2-2$.*

Thus the whole failure set of (2) on the orthant lies in the simplex $\{d\ge 0: d_1+d_2+d_3<2\sqrt2-2\}$, a region that contains no lattice point other than the origin, where $F=0$. The proof is two polynomial identities (Section 3). The lattice inequality (2) follows at once, and we also give it a direct uniform proof by a single substitution (Proposition 3.3). That matters because the case analysis of [1] is defective in two of its three branches: in the branch $d_i\ge 2$ a printed rewriting of one of the three partial sums equals -15 at $(5,4,0)$ and -2117 at $(20,19,2)$, where it is asserted to be non-negative, and the $d_3=1$ display is not equal to $F(d_1,d_2,1)$ at all (Section 4). Both defects have exact repairs, which we state and prove. A third identity of the same proof, which an earlier reading had reported as false, is correct once the summation convention is fixed (Proposition 4.3). None of this affects the conclusion of Proposition 3.1 or Theorem 3.2 of [1]: inequality (2) is true.

Two smaller items complete the audit of the proof of Hunter's conjecture. The induction step of Theorem 3.2 of [1], in the case $\gamma_1\le\gamma_2+\gamma_3$, uses the inequality $a^2+b^2\ge 2/n$, introduced there by "we obtain that" and not proved; we prove it in a weighted real form (Section 5). And the source's partial treatment of the case $n=2$ of its real-exponent conjecture asserts three algebraic facts about a generalised polynomial g; we certify them and record that one of them, the count of five sign changes, needs the hypothesis $q > 2$ (Section 6).

Every statement in Sections 3–6 is machine-checked in Lean 4 over Mathlib; Section 8 and Appendix A give the details. Section 7 is different in kind: it summarises, without proof and without formal backing, what is known about the real-exponent conjecture beyond the range settled in the literature, and describes planned work.

The real-exponent conjecture. Section 4 of [1] turns to non-negative coefficients. Its Definition 4.4 sets, for i.i.d. standard exponentials $X_1, X_2, \dots$ and real q,

$$\rho(j,q) := \mathbb{E}\Big(\frac{X_1 + \cdots + X_j}{\sqrt{j}}\Big)^q, \qquad \text{so that} \qquad \rho(j,q) = \frac{\Gamma(j+q)}{j^{q/2}\,\Gamma(j)},$$

since $X_1 + \cdots + X_j$ has the Gamma distribution with shape j. Its Theorem 4.5 proves that for every non-negative *integer* k and non-negative $a_1, \dots, a_n$ with $\sum a_i^2 = 1$, $\mathbb{E}(a_1X_1 + \cdots + a_nX_n)^k \geq \min\{\rho(1,k), \dots, \rho(n,k)\}$, and its Conjecture 4.8 reads:

> *Let $q > 0$ be a real number, and let $a_1, \dots, a_n$ be non-negative real numbers satisfying $\sum_{i=1}^n a_i^2 = 1$. If $X_1, \dots, X_n$ are i.i.d. standard exponential random variables, then*
> $$\mathbb{E}\big(a_1X_1 + \cdots + a_nX_n\big)^q \;\geq\; \min\{\rho(1,q), \dots, \rho(n,q)\}.$$

Brazitikos, Tang and Tkocz [2] settle this for $q \leq 4$. For $x_1, \dots, x_n \geq 0$ let $M_p(x_1, \dots, x_n) = \mathbb{E}\big[\big(\sum_j \sqrt{x_j}\,\mathcal{E}_j\big)^p\big]$ with $\mathcal{E}_j$ i.i.d. standard exponentials (their equation (7)). Their Theorem 5 states: "For an arbitrary integer $n \geq 2$, function M_p is Schur-convex for $-1 < p < 0$, and Schur-concave for $0 \leq p \leq 4$ on $\mathbb{R}^n_+$. Moreover, this function fails to be Schur-monotone as long as $p > 4$." Since every vector $(a_1^2, \dots, a_n^2)$ with non-negative entries summing to 1 is majorised by $(1, 0, \dots, 0)$, Schur-concavity gives $\mathbb{E}(\sum_i a_iX_i)^q = M_q(a_1^2, \dots, a_n^2) \geq M_q(1, 0, \dots, 0) = \rho(1,q)$ for $0 \leq q \leq 4$, which is Conjecture 4.8 for every n and every $q \in (0, 4]$. The open range is therefore $q > 4$. The source itself offers, as support, its Proposition 4.12, the case $n = 2$ for all real $q \geq 0$; its argument for $q \geq 4$ establishes three algebraic facts about the numerator g of a derivative and ends without drawing a conclusion from them (Section 6). Remark 2 of [2] states the real-p form of Hunter's conjecture itself, for a fixed even n and unrestricted signs, as open; that is a different problem and nothing here touches it.

2. Notation

Definition 2.1. For a commutative ring R, a shift $s \in R$ and $d = (d_1, d_2, d_3) \in R^3$ put

$$\begin{aligned} F_s(d_1,d_2,d_3) = {} & (d_1+s)^2(d_1+d_2-d_3)(d_1+d_3-d_2) \\ & + (d_2+s)^2(d_2+d_3-d_1)(d_2+d_1-d_3) \\ & + (d_3+s)^2(d_3+d_1-d_2)(d_3+d_2-d_1), \end{aligned}$$

and $F := F_2$. (In the formal development these are `hunterFs` and `hunterF`.)

The i-th summand of F is $(d_i+2)^2\big(d_i^2 - (d_j - d_k)^2\big)$, so F is symmetric under all permutations of (d_1, d_2, d_3), a fact the source does not use. It lets us sort the coordinates: given $d_1 \geq d_2 \geq d_3$ we write

$$x = d_1 - d_2, \qquad y = d_2 - d_3, \qquad z = d_3, \tag{3}$$

so that $(d_1, d_2, d_3) = (x+y+z,\ y+z,\ z)$ and $s := d_1 + d_2 + d_3 = x + 2y + 3z$.

3. The lattice inequality and its exact real relaxation

Theorem 3.1. *Let $d_1, d_2, d_3 \geq 0$ be real and $s = d_1 + d_2 + d_3$.*

(i) *If $s^2 + 4s \geq 4$ then $F(d_1, d_2, d_3) \geq 0$.*
(ii) *For $s \geq 0$, $s^2 + 4s \geq 4$ if and only if $s \geq 2\sqrt{2} - 2$; indeed $(2\sqrt{2}-2)^2 + 4(2\sqrt{2}-2) = 4$.*
(iii) *(Sharpness.) If $t > 0$ and $t^2 + 4t < 4$ then $F(t, 0, 0) < 0$.*

Consequently $F \geq 0$ on $\{d \in \mathbb{R}^3_{\geq 0} : d_1 + d_2 + d_3 \geq 2\sqrt{2} - 2\}$, and this threshold is the best possible constant of its kind.

Proof. By the symmetry of F it suffices to treat $d_1 \geq d_2 \geq d_3 \geq 0$, and we use the coordinates (3), so $x, y, z \geq 0$ and $s = x + 2y + 3z$. The proof rests on two identities in $\mathbb{Z}[x, y, z]$, both verified by normalisation in the polynomial ring (`ring`):

$$F(x+y+z,\, y+z,\, z) = x^2\,(s^2+4s-4) + z\,K, \tag{4}$$
$$K = 2x(4-4x-x^2) + z(12+12x-5x^2) + y(16+16x-2x^2) + N,$$
$$N = 3z^3 + 8yz^2 + 10y^2z + 4y^3 + 4xz^2 + 10xyz + 6xy^2 + 12z^2 + 24yz + 16y^2,$$

and

$$F(x+y+z,\, y+z,\, z) = x^2\,(x^2+4x-4) + M, \tag{5}$$
$$\begin{aligned} M &= x^2(4z^2+10yz+4y^2+4z+8y+4xz+4xy) \\ &\quad + \big(3z^4+8yz^3+10y^2z^2+4y^3z+4xz^3+10xyz^2+6xy^2z+12z^3+24yz^2 \\ &\qquad + 16y^2z+12xz^2+16xyz+12z^2+16yz+8xz\big). \end{aligned}$$

Every monomial of N and of M has a non-negative coefficient, so $N, M \geq 0$ on the orthant. If $x^2+4x \geq 4$, (5) gives $F \geq 0$ with no hypothesis on s. Otherwise $x^2+4x \leq 4$, hence $x \leq 1$, hence $4-4x-x^2 \geq 0$, $12+12x-5x^2 \geq 0$ and $16+16x-2x^2 \geq 0$, so $K \geq 0$; and the hypothesis $s^2+4s \geq 4$ makes the first term of (4) non-negative. This proves (i).

For (ii), $s^2+4s-4 = (s+2)^2-8$ is increasing on $s \geq 0$ and vanishes at $s = 2\sqrt{2}-2$. For (iii), $F(t,0,0) = t^2(t^2+4t-4)$ by Definition 2.1, and the second factor is negative exactly when $t^2+4t < 4$. ☐

The hypothesis of (i) is deliberately stated as the polynomial inequality $s^2+4s \geq 4$, which keeps the statement and the sharpness witnesses inside rational arithmetic; (ii) converts it into the threshold $2\sqrt{2}-2 = 0.8284\ldots$.

Corollary 3.2. *If $d_1, d_2, d_3 \geq 0$ are real with $d_1+d_2+d_3 \geq 1$, then $F(d_1, d_2, d_3) \geq 0$.*

Proof. $s \geq 1$ gives $s^2+4s \geq 5 > 4$. ☐

In particular (2) holds: a non-zero point of $\mathbb{N}^3$ has coordinate sum at least 1, and $F(0,0,0) = 0$. The next proposition proves (2) again, directly over the integers and without passing through the reals; this is the certificate that replaces the case analysis of [1].

Proposition 3.3. $F(d_1, d_2, d_3) \geq 0$ *for every* $(d_1, d_2, d_3) \in \mathbb{N}^3$.

Proof. By symmetry assume $d_1 \geq d_2 \geq d_3$ and substitute (3) with $x, y, z \in \mathbb{N}$. The expansion of $F(x+y+z, y+z, z)$ has exactly one negative coefficient, that of x^2, and splitting on whether $x = 0$ or $x = w+1$ removes it: in $\mathbb{Z}[y, z]$ and $\mathbb{Z}[w, y, z]$ respectively,

$$F(y+z,\, y+z,\, z) = 3z^4+8yz^3+10y^2z^2+4y^3z+12z^3+24yz^2+16y^2z+12z^2+16yz,$$
$$\begin{aligned} F(w+1+y+z,\, y+z,\, z) &= 3z^4+8yz^3+10y^2z^2+4y^3z+4wz^3+10wyz^2 \\ &\quad + 6wy^2z+4w^2z^2+10w^2yz+4w^2y^2+4w^3z+4w^3y+w^4+16z^3 \\ &\quad + 34yz^2+22y^2z+20wz^2+36wyz+8wy^2+16w^2z+20w^2y+8w^3 \\ &\quad + 28z^2+42yz+4y^2+28wz+28wy+14w^2+16z+12y+8w+1. \end{aligned}$$

Both right-hand sides have non-negative integer coefficients, so each is the value of a polynomial with coefficients in $\mathbb{N}$ at a point of $\mathbb{N}^3$, hence non-negative. ☐

The proposition is stated in $\mathbb{Z}$, with the d_i cast from $\mathbb{N}$; its formal proof uses no decision procedure and no case enumeration, only the two identities and the fact that a natural number is non-negative. A census over $0 \leq d_3 \leq d_2 \leq d_1 \leq 120$ found no negative value before the certificate was written; nothing rests on that census.

Proposition 3.4. *In every commutative ring,*

$$F(d_1, d_2, 0) = (d_1-d_2)^2\big((d_1+d_2)^2+4(d_1+d_2)-4\big). \tag{6}$$

Over the reals, $F(\frac{1}{2}, 0, 0) = -\frac{7}{16}$; hence it is not the case that $F(d_1, d_2, d_3) \geq 0$ for all real $d_1, d_2, d_3 \geq 0$.

Proof. (6) is a polynomial identity, and the evaluation is direct. □

Identity (6) is the $d_3 = 0$ branch of the proof in [1], printed there as "$(d_1 - d_2)^2(d_1^2 + 2d_1d_2 + d_2^2 + 4d_1 + 4d_2 - 4) \geq 0$, which is true." It is true for integers, because a non-zero integer sum $d_1 + d_2$ is at least 1; over the reals the second factor is negative exactly for $0 < d_1 + d_2 < 2\sqrt{2} - 2$. By Corollary 3.2 the other two branches of the case analysis, $d_3 = 1$ and $d_i \geq 2$, are true for real d_i as well, so this is the single place in the proof of (2) where the integrality of the multiplicities γ_i is used, and the source does not mention it. Since $F \geq 0$ is false on part of the orthant, no certificate of non-negativity of F over the whole orthant — in particular no sum-of-squares representation with non-negative multipliers — can exist; Theorem 3.1 gives the correct real domain.

Proposition 3.5 (controls for Theorem 3.1)**.** (i) *The threshold hypothesis cannot be dropped: at* $(\frac{1}{2}, 0, 0)$ *one has* $s^2 + 4s = \frac{9}{4} < 4$ *and* $F = -\frac{7}{16}$*.*
(ii) *The conclusion cannot be strengthened to* $F > 0$*:* $F(1, 1, 0) = 0$ *while* $s = 2$ *satisfies* $s^2 + 4s \geq 4$*.*
(iii) *The constant brackets* $2\sqrt{2} - 2$*:* $F(\frac{9}{10}, 0, 0) > 0$ *and* $F(\frac{8}{10}, 0, 0) < 0$*.*

Proposition 3.6 (controls for Proposition 3.3)**.** (i) $F(0, 0, 0) = 0$*,* $F(1, 1, 0) = 0$*,* $F(1, 0, 0) = 1$ *and* $F(2, 1, 1) = 64$*; in particular it is not the case that* $F(d) \geq 1$ *for all* $d \in \mathbb{N}^3$*.*
(ii) *The shift* 2 *in* $(d_i+2)^2$ *is load-bearing: with shift* 3 *the lattice statement already fails,* $F_3(1, 0, 0) = -2$*, so it is not the case that* $F_3(d) \geq 0$ *for all* $d \in \mathbb{N}^3$*.*

Both propositions are evaluations of F at explicit points. For completeness we also record the other algebraic fact of the base case, the leading-coefficient identity, which, unlike (2), is a genuine real statement.

Proposition 3.7. *Identity* (1) *holds in every commutative ring, and for real* $d_1, d_2, d_3 \geq 0$ *its left-hand side is non-negative.*

Proof. The identity is a polynomial identity; the right-hand side is manifestly non-negative on the orthant. □

4. Two errata in the proof of Proposition 3.1 of [1]

The branch $d_1, d_2, d_3 \geq 2$ of the proof of Proposition 3.1 in [1] reads (verbatim, with the sorting assumption $\gamma_1 \geq \gamma_2 \geq \gamma_3$ in force):

> *Finally, if* $d_1, d_2, d_3 \geq 2$*, we rewrite the inequality as*
> $$\sum(d_1+2)^2(d_1-d_3)(d_1-d_2) + \sum(d_1+2)^2 d_3(d_1-d_3) + \sum(d_1+2)^2 d_2 d_3 \;\geq\; 0.$$
> *The last sum is clearly non-negative. For the first sum notice that it can be expressed as*
> $$(d_1-d_2)\big[(d_1+2)^2(d_1-d_2) - (d_2+2)^2(d_2-d_3)\big] + (d_3+2)^2(d_3-d_2)(d_3-d_1) \;\geq\; 0$$
> *since* $d_1 \geq d_2 \geq d_3$*. For the second one, after collecting the same terms, equals to*
> $$\sum(d_1-d_3)^2(d_1d_3-4) \;\geq\; 0,$$
> *which is again true, since* $d_1, d_2, d_3 \geq 2$*.*

4.1. The printed bracket.

Let

$$\begin{aligned}
S_1(d) &= (d_1+2)^2(d_1-d_3)(d_1-d_2) + (d_2+2)^2(d_2-d_1)(d_2-d_3) + (d_3+2)^2(d_3-d_2)(d_3-d_1),\\
B_{\mathrm{pr}}(d) &= (d_1-d_2)\big[(d_1+2)^2(d_1-d_2) - (d_2+2)^2(d_2-d_3)\big] + (d_3+2)^2(d_3-d_2)(d_3-d_1),\\
B_{\mathrm{c}}(d) &= (d_1-d_2)\big[(d_1+2)^2(d_1-d_3) - (d_2+2)^2(d_2-d_3)\big] + (d_3+2)^2(d_3-d_2)(d_3-d_1),
\end{aligned}$$

so that S_1 is the first sum of the display, B_{pr} is its printed rewriting, and B_{c} differs from B_{pr} in a single factor, $(d_1 - d_3)$ in place of the first $(d_1 - d_2)$ inside the bracket.

Theorem 4.1. (i) $B_{\mathrm{c}}(d) = S_1(d)$ *in every commutative ring.*
(ii) $B_{\mathrm{pr}}(5, 4, 0) = -15$*,* $B_{\mathrm{pr}}(20, 19, 2) = -2117$*, while* $S_1(20, 19, 2) = 6111$*.*
(iii) *For real* $d_1 \geq d_2 \geq d_3 \geq 0$*,* $S_1(d) \geq 0$*.*

Proof. (i) is a polynomial identity and (ii) is evaluation. For (iii) use (i): under $d_1 \ge d_2 \ge d_3 \ge 0$ we have $(d_2+2)^2 \le (d_1+2)^2$ and $0 \le d_2 - d_3 \le d_1 - d_3$, so the bracket in B_{c} is non-negative and so is its prefactor $d_1 - d_2$; and $(d_3+2)^2(d_3-d_2)(d_3-d_1) = (d_3+2)^2(d_2-d_3)(d_1-d_3) \ge 0$. □

So the expression printed in [1] is not a rewriting of the first sum, and it is not non-negative under the stated hypothesis: the point $(20,19,2)$ lies in the branch $d_i \ge 2$ where the display is used, and $(5,4,0)$ is the smallest sorted lattice point at which the printed expression is negative: an exhaustive evaluation of B_{pr} over $d_1 \ge d_2 \ge d_3 \ge 0$ with $d_1 \le 40$, which is not part of the formal development, finds no negative value at any point of coordinate sum less than 9, nor at any point that precedes $(5,4,0)$ in lexicographic order. With the one-letter repair the sentence of [1] becomes correct, and correct over the reals.

4.2. **The $d_3 = 1$ display.** The branch $d_3 = 1$ of the same proof reads: "If $d_3 = 1$, then the inequality is equivalent to

$$(d_1^2-d_2^2)^2+(d_1+d_2)(d_1-d_2)^2+(2d_1^3-10d_1^2+4d_1+1+17d_1d_2)+(2d_2^3-10d_2^2+4d_2+1+17d_1d_2) \;\ge\; 0,$$

which holds." Write $D_{\mathrm{pr}}(d_1,d_2)$ for the displayed expression and

$$D_{\mathrm{c}}(d_1,d_2) = (d_1^2-d_2^2)^2+2(d_1+d_2)(d_1-d_2)^2+(2d_1^3-10d_1^2+4d_1+1+17d_1d_2)+(2d_2^3-10d_2^2+4d_2+17d_1d_2)$$

for the same grouping with the middle term doubled and one of the two constants $+1$ removed.

Theorem 4.2. *In every commutative ring,*

$$F(d_1,d_2,1) - D_{\mathrm{pr}}(d_1,d_2) = (d_1-d_2)^2(d_1+d_2) - 1 \qquad \textit{and} \qquad F(d_1,d_2,1) = D_{\mathrm{c}}(d_1,d_2).$$

In particular $F(2,1,1) = 64$ *whereas* $D_{\mathrm{pr}}(2,1) = 62$.

Proof. Both are polynomial identities; the values are evaluations. □

The displayed expression is therefore not equivalent to the $d_3 = 1$ case of (2), although the difference is harmless for the conclusion of [1]: an elementary estimate, which we have not formalised, shows that $D_{\mathrm{pr}} \ge 28$ on $d_1 \ge d_2 \ge 1$, so that $F(d_1,d_2,1) \ge 27$ there, and in any event (2) holds by Proposition 3.3.

4.3. **The summation convention.** The last sentence of the quoted branch asserts that the second sum, after collecting terms, equals $\sum(d_1-d_3)^2(d_1d_3-4)$. The three $\sum$ signs of the display carry no index set. Read as sums over the distinct terms obtained by permuting the indices $1,2,3$ — the usual convention for symmetric sums — the first and third summands, which are symmetric under exchanging the two indices other than the leading one, produce three terms each, while the summand $(d_1+2)^2d_3(d_1-d_3)$, which has no such symmetry, produces six, and $(d_1-d_3)^2(d_1d_3-4)$ produces three. Set

$$\begin{aligned}
T_{\mathrm{ord}}(d) &= \sum_{i\neq j}(d_i+2)^2d_j(d_i-d_j) && \text{(six terms)},\\
R_{\mathrm{un}}(d) &= \sum_{i<j}(d_i-d_j)^2(d_id_j-4) && \text{(three terms)},\\
T_{\mathrm{cyc}}(d) &= \sum_{\mathrm{cyc}}(d_i+2)^2d_k(d_i-d_k) && \text{(three terms)},\\
R_{\mathrm{cyc}}(d) &= \sum_{\mathrm{cyc}}(d_i-d_k)^2(d_id_k-4) && \text{(three terms)},
\end{aligned}$$

where the cyclic sums run over $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$, and let $S_3(d) = \sum_{\mathrm{cyc}}(d_i+2)^2d_jd_k$.

Proposition 4.3. (i) $T_{\mathrm{ord}}(d) = R_{\mathrm{un}}(d)$ *in every commutative ring.*
(ii) $T_{\mathrm{cyc}}(3,2,2) = 2$ *but* $R_{\mathrm{cyc}}(3,2,2) = 4$*, and* $T_{\mathrm{cyc}}(4,3,2) = -4$ *but* $R_{\mathrm{cyc}}(4,3,2) = 26$.
(iii) $F(d) = S_1(d) + T_{\mathrm{ord}}(d) + S_3(d)$ *in every commutative ring.*

Proof. All three are polynomial identities or evaluations. Behind (i) is the pairing of the terms (i,j) and (j,i) of T_{ord}: $(d_i+2)^2d_j - (d_j+2)^2d_i = (d_i-d_j)(d_id_j-4)$. □

Thus the identity of [1] is correct under the distinct-permutation reading, and so is the three-sum splitting on which the branch is built; under a uniform cyclic reading of all three $\sum$ signs the identity is false. We record this because an earlier reading of the proof, made in the course of this work, took the sums cyclically and reported the identity as a further error; it is not one, though the display does invite the misreading, since its three sums range over index sets of different sizes.

5. The step $a^2 + b^2 \geq 2/n$

In the proof of Theorem 3.2 of [1], in the case $\gamma_1 \leq \gamma_2+\gamma_3$, the induction step is carried by the inequality $a^2 + b^2 \geq 2/n$, where a, b, c are the three coordinate values of an extremum ordered by $a > |b| > |c|$, with multiplicities $\gamma_1, \gamma_2, \gamma_3 \geq 1$ summing to n and $\gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1$. The source writes "In this case, we obtain that $a^2 + b^2 \geq \frac{2}{n}$, which helps us to complete the induction" and gives no argument. The inequality is true, and the argument never uses that the multiplicities are integers.

Proposition 5.1. (i) *Let $\gamma_1, \gamma_2, \gamma_3 > 0$ be real with $\gamma_1 \leq \gamma_2 + \gamma_3$, and let $u \geq v \geq w \geq 0$ be real with $\gamma_1 u + \gamma_2 v + \gamma_3 w = 1$. Then $u + v \geq 2/(\gamma_1 + \gamma_2 + \gamma_3)$.*

(ii) *In the source's variables: if $\gamma_1, \gamma_2, \gamma_3 \geq 1$ are integers with $\gamma_1 + \gamma_2 + \gamma_3 = n$ and $\gamma_1 \leq \gamma_2 + \gamma_3$, and a, b, c are real with $a^2 \geq b^2 \geq c^2$ and $\gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1$, then $a^2 + b^2 \geq 2/n$.*

Proof. (i) Since $w \leq v$, $1 = \gamma_1 u + \gamma_2 v + \gamma_3 w \leq \gamma_1 u + (\gamma_2 + \gamma_3)v$. Since $v \leq (u + v)/2$ and $\gamma_2 + \gamma_3 - \gamma_1 \geq 0$,

$$\gamma_1 u + (\gamma_2 + \gamma_3)v = \gamma_1(u + v) + (\gamma_2 + \gamma_3 - \gamma_1)v \leq \gamma_1(u + v) + (\gamma_2 + \gamma_3 - \gamma_1)\tfrac{u+v}{2} = \tfrac{u+v}{2}(\gamma_1 + \gamma_2 + \gamma_3).$$

(ii) is (i) with $(u, v, w) = (a^2, b^2, c^2)$. □

Proposition 5.2 (controls for Proposition 5.1)**.** (i) *The hypothesis $\gamma_1 \leq \gamma_2 + \gamma_3$ cannot be dropped: at $\gamma = (4, 1, 1)$, $(u, v, w) = (\frac{1}{4}, 0, 0)$ the remaining hypotheses of (i) hold, but $u + v = \frac{1}{4} < \frac{1}{3}$.*

(ii) *The ordering $w \leq v$ cannot be dropped: at $\gamma = (1, 1, 1)$, $(u, v, w) = (\frac{1}{2}, 0, \frac{1}{2})$ the remaining hypotheses hold, but $u + v = \frac{1}{2} < \frac{2}{3}$.*

(iii) *The constant is best possible: with $\gamma = (1, 1, 1)$ and $u = v = w = \frac{1}{3}$ all hypotheses hold with $u + v = \frac{2}{3} = 2/(\gamma_1 + \gamma_2 + \gamma_3)$, so the conclusion cannot be made strict.*

6. The case $n = 2$ of Conjecture 4.8: the algebra of Proposition 4.12

Proposition 4.12 of [1] states: "Let q be a non-negative real number and $a, b \geq 0$ such that $a^2 + b^2 = 1$. Then $\mathbb{E}(aX + bY)^q \geq \min\{\mathbb{E}\big(\frac{X+Y}{\sqrt{2}}\big)^q, \mathbb{E}(X_1^q)\}$." For $q \leq 4$ its proof applies a log-concavity lemma (Lemma 4.13 of [1], for which [1] refers to [5]). For $q \geq 4$ it sets $x = a/b$, $f(x) = \log\big(\frac{x^{q+1}-1}{x-1}\big) - \frac{q}{2}\log(x^2+1)$, and computes

$$f'(x) = \frac{-x^{q+2} + qx^{q+1} - (q + 1)x^q + (q + 1)x^2 - qx + 1}{(x - 1)(x^2 + 1)(x^{q+1} - 1)}.$$

It then asserts, verbatim: "Note that the numerator, say g, has five sign changes in its coefficients, therefore by the extension of Descartes' rule of signs [4] it has at most five positive roots, and thus the same will hold for f'. It is easy to check that $g(x)$ has a root of multiplicity three at $x = 1$. We observe that $g'''(1) = q(q + 1)(q - 4) > 0$ for $q > 4$ and thus from the higher order derivative test, $x = 1$ is a saddle point and a strictly increasing point of inflection." The proof ends with that sentence. We do not complete the argument here; we certify the algebra of the three assertions and record where the first one needs a hypothesis.

For real q and $x > 0$ put

$$g_q(x) = -x^{q+2} + q\,x^{q+1} - (q + 1)\,x^q + (q + 1)\,x^2 - q\,x + 1,$$
$$A(m, q) = -(q + 2)^m + q(q + 1)^m - (q + 1)q^m + (q + 1)2^m - q \qquad (m \in \mathbb{N}).$$

Proposition 6.1. *For every real q:*

(i) $g_q(1) = 0$.

(ii) *For every real t, $g_q(e^t) = -e^{(q+2)t} + q\,e^{(q+1)t} - (q + 1)\,e^{qt} + (q + 1)\,e^{2t} - q\,e^t + 1$.*

(iii) $A(0, q) = -1$, $A(1, q) = A(2, q) = 0$ *and* $A(3, q) = q(q + 1)(q - 4)$.

(iv) *If $q > 0$ then $A(3, q) > 0$ if and only if $q > 4$.*

(v) $A(3,5) = 30$, $A(3,2) = -12$ *and* $g_5(2) = 15$.

Proof. (i), (iii) and (v) are direct; (ii) is $(e^t)^c = e^{ct}$; (iv) is the sign of $q(q+1)(q-4)$ for $q > 0$. □

By (ii), $g_q(e^t) = \sum_{m \geq 0} c_m t^m/m!$ with $c_0 = A(0,q) + 1 = 0$ and $c_m = A(m,q)$ for $m \geq 1$. Since $t \mapsto e^t$ is a local diffeomorphism at $t = 0$ with $e^0 = 1$, the vanishing of c_0, c_1, c_2 is the vanishing of g_q, g_q', g_q'' at $x = 1$, and then $c_3 = g_q'''(1)$; so (iii) is the source's "root of multiplicity three at $x = 1$" together with its value $g'''(1) = q(q+1)(q-4)$, and (iv) is the reason the argument of [1] starts at $q = 4$. This passage from the coefficients of the exponential sum to the derivatives of g_q is the standard power-series argument and is not part of the formal development; (iii) is stated and checked as an algebraic identity.

For the sign count, order the six monomials of g_q by exponent. For $q > 2$ the exponents satisfy $0 < 1 < 2 < q < q+1 < q+2$ and the coefficient sequence is $(1, -q,\ q+1,\ -(q+1),\ q,\ -1)$; for $1 < q < 2$ they satisfy $0 < 1 < q < 2 < q+1 < q+2$ and the sequence is $(1, -q,\ -(q+1),\ q+1,\ q,\ -1)$. Let sgn be the sign function and let ch count the sign changes of a finite sequence, i.e. the number of consecutive pairs with negative product.

Proposition 6.2. (i) *If $q > 2$, the sign sequence of the coefficients of g_q ordered by exponent is $(+,-,+,-,+,-)$, with five sign changes.*
(ii) *If $1 < q < 2$, it is $(+,-,-,+,+,-)$, with three sign changes.*
(iii) *The two counts differ.*

Proof. The exponent orders are the stated chains of inequalities; the signs of $1, -q, \pm(q+1), q, -1$ are immediate; ch of a six-element list is a finite evaluation. □

So the unqualified sentence "the numerator, say g, has five sign changes in its coefficients" requires $q > 2$: for $1 < q < 2$ the exponents 2 and q change places and the count is three. This is a qualification rather than an error — [1] uses the count only for $q \geq 4$ — but any argument that starts from the sign count must carry the hypothesis.

7. The real-exponent conjecture beyond $q = 4$: status and planned work

Nothing in this section is machine-checked, and nothing in Sections 3–6 depends on it. We record what is known about Conjecture 4.8 of [1] for $q > 4$, in the form in which it stands in our working records, with its status marked in each case. Throughout, $X_1, \ldots, X_n$ are i.i.d. standard exponentials, $a \in \mathbb{R}^n_{\geq 0}$ is a unit vector, $S_a = \sum_i a_i X_i$, and $r_j(q) = \rho(j,q)/\Gamma(1+q)$, so that $r_1 \equiv 1$ and $r_2(q) = (1+q)2^{-q/2}$.

Remark 7.1 (the range $[0,5]$; argued in prose, not formalised). The conjecture holds for every n and every real $q \in (0,5]$. The argument: $R(q) := \min_j r_j(q)$ equals 1 for $q \leq q^* = 5.3197223558\ldots$, the positive root of $1 + q = 2^{q/2}$, because $r_1 \equiv 1$ and $r_2(q) = (1+q)2^{-q/2}$ is the first of the r_j to drop below 1 (a numerical statement, re-checked for this paper: the point at which r_j first falls below 1 is q^* for $j = 2$ and increases with j for $2 \leq j \leq 40$, and $\min_{j \leq 300} r_j = 1$ on $[0, q^*]$); so on that range the conjecture is $G_a(q) := \mathbb{E}S_a^q/\Gamma(1+q) \geq 1$. Now S_a has a log-concave density, so G_a is log-concave on $(-1, \infty)$ by Lemma 4.13 of [1]; $G_a(0) = 1$; and $G_a(5) \geq 1$ is the integer case, Theorem 4.5 of [1] at $k = 5$. Log-concavity between 0 and 5 gives $G_a \geq 1$ on $[0,5]$. Of this range, $(0,4]$ is Theorem 5 of [2]; only the increment $(4,5]$ is new, and it is marginal. The route is capped at $q = 5$: numerically the chord bound falls short of $\min_j \rho(j,q)$ by a factor 0.9582 near $q \approx 5.32$. The log-concavity lemma is not available in Mathlib, which is why this remains prose.

Remark 7.2 (two-value reduction; argued in prose, not formalised). For real $p > 2$, every minimiser of $a \mapsto \mathbb{E}S_a^p$ on the non-negative part of the unit sphere takes at most two distinct values on its support. The argument works on the relative interior of the face spanned by the support, where the Lagrange conditions hold with equality, and uses the difference identity of Lemma 7 of [2] together with their equation (8), $p\,\mathbb{E}\Phi(S_x) = \mathbb{E}\Phi''(S_x + x_1\mathcal{E} + x_2\mathcal{E}')$ at a critical point with $x_1 \neq x_2$, for $\Phi(t) = t^p$: three distinct values on the support would force $\mathbb{E}\Phi'''(S_a + a\mathcal{E} + b\mathcal{E}' + c\mathcal{E}'') = 0$, impossible since $\Phi'''(t) = p(p-1)(p-2)t^{p-3} > 0$. Zero coordinates are not covered, because on the boundary the multiplier condition is an inequality. So every minimiser is either a flat vector on its support, whose moment is some $\rho(j,q)$ outright, or a two-block vector $a = (\alpha^{\gamma_1}, \beta^{\gamma_2}, 0^{n-m})$ with $\alpha \neq \beta$ and $m = \gamma_1 + \gamma_2 \leq n$.

Remark 7.3 (conditional result for $q > n+2$; prose, and conditional)*.* For a two-block vector with $x = \alpha/\beta$, the function $\Psi(x) = \log \mathbb{E}(xG_1 + G_2)^q - \frac{q}{2}\log(\gamma_1 x^2 + \gamma_2)$, with G_1, G_2 independent Gamma variables of shapes γ_1, γ_2, takes the values $\log\rho(\gamma_2, q)$, $\log\rho(m, q)$, $\log\rho(\gamma_1, q)$ at $x = 0, 1, \infty$, and the conjecture for n is equivalent to Ψ being minimised at one of these three points for every (γ_1, γ_2) with $\gamma_1 + \gamma_2 \le n$. The critical-point equation has the form $N(x) = x^q P(x) + Q(x) = 0$ with polynomials P, Q whose coefficients are polynomials in q; at $(\gamma_1, \gamma_2) = (1, 1)$, N is the numerator g of Section 6. Two claims are conjectural: *Claim A*, that for $q > 2$ the coefficient sequence of N, merged in exponent order, has exactly $m + 3$ sign changes (equivalently, that all $m + 4$ coefficients are non-zero and strictly alternate); and *Claim B*, that N vanishes to order exactly $m + 1$ at $x = 1$ when $q \ne m + 2$, equivalently $\Psi''(1) = q\gamma_1\gamma_2(q - m - 2)/(m^2(m + 1))$. Conditional on Claims A and B, the conjecture holds for every n and every real $q > n + 2$; combined with Remark 7.1, for all real q when $n \le 3$; and for each n the residual is the compact window $q \in (5, n + 2]$. Claim A has been tested on 14,713 exact-rational triples (γ_1, γ_2, q) with $m \le 24$ (11,401 of them with $q > 2$) with no deviation, and its coefficient half symbolically in q for all 435 pairs with $m \le 30$, where $\deg P = m$ and $\deg Q = \gamma_2 + 1$ exactly; Claim B has been checked numerically at six points only. The deduction from the sign count to a root count is a rule of signs for generalised polynomials with real exponents; its ordinary polynomial form is in Mathlib (`Polynomial.signVariations` and `Polynomial.roots_countP_pos_le_signVariations`, roots counted with multiplicity), and the real-exponent form is not needed, since for rational $q = a/b$ the substitution $x = y^b$ turns N into a polynomial with the same coefficient sequence and the same positive roots with multiplicity, and the conjectured inequality is closed in q.

Remark 7.4 (planned work: the window $q \in (5, n + 2]$)*.* The threshold in Remark 7.3 is $q > m + 2$ for a minimiser of support size m, so for a fixed n the open set is finite and small: nothing for $n \le 3$; for $n = 4$ the two pairs $(\gamma_1, \gamma_2) \in \{(1, 3), (2, 2)\}$ on $q \in (5, 6]$; for $n = 5$ additionally $(1, 4), (2, 3)$ on $q \in (5, 7]$. The planned attack on $n = 4$ is a certified interval/Bernstein computation on the compact boxes $x \in (0, 1]$, $q \in [5, 6]$ (the duality $\Psi_{\gamma_1,\gamma_2}(x) = \Psi_{\gamma_2,\gamma_1}(1/x)$ folds $x > 1$ back), in a Gamma-free form obtained from $\Gamma(m + q)/\Gamma(j + q) = \prod_{i=j}^{m-1}(i + q)$, with the strip near $x = 1$ handled by a Taylor expansion because the inequality is an identity at $x = 1$. The planned dispatch — this certificate together with the rational-exponent lemma and a symbolic verification of Claim B — is priced at 900–1400 lines of Lean and at most two CPU-hours, and it would give Conjecture 4.8 for $n = 4$ and every real $q > 0$. Two shortcuts were tested numerically and are dead: "no critical point of Ψ other than $x = 1$ when $q < m + 2$" is false (2736 of 13,125 window cells examined have three critical points), and an induction $q \mapsto q - 2$ through a critical-point recursion, with the endpoint bound at $q - 2$, fails at every one of 3752 interior critical points found. None of this is a result of the present paper.

8. Verification

Every statement of Sections 3–6 — Theorems 3.1, 4.1, 4.2, Corollary 3.2 and Propositions 3.3–3.7, 4.3, 5.1, 5.2, 6.1 and 6.2 — is formalised and checked in Lean 4 [6] (toolchain v4.32.0) over Mathlib [7], in four modules of about 850 lines in total; Appendix A reproduces the exact statement of each formal declaration and the axioms it depends on, as reported by `#print axioms`. The polynomial identities are closed by normalisation in the commutative ring (`ring`), and are therefore proved over an arbitrary commutative ring, as stated; the real inequalities by `positivity`, `nlinarith` and `linarith`; the numerical witnesses by `norm_num`; the two six-element sign-change counts of Proposition 6.2 by kernel evaluation (`decide`). The development contains no `sorry`, declares no axiom of its own and does not use `native_decide`; the identities and integer evaluations depend only on `propext` and `Quot.sound`, the statements over $\mathbb{R}$ additionally on `Classical.choice` through Mathlib's construction of the reals. No result rests on an exhaustive computation: the lattice census to $d_1 \le 120$ and the grid search that first located the real failure region (58,156 failure points at step 0.0038, all with $d_1 + d_2 + d_3 \le 0.8270$) were exploratory, and Theorem 3.1 supersedes them. Not formalised are the reduction of Proposition 3.1 of [1] to (1) and (2), which we take from the source, the estimate $D_{\mathrm{pr}} \ge 28$ mentioned after Theorem 4.2, the power-series step after Proposition 6.1, and everything in Section 7. The repository is private; repository reference to be added at submission.

Appendix A. The formal statements

The declarations below live in the namespace `LeanProblemSpec.BrazitikosRealq`, in the modules `H4Lattice`, `RealThreshold`, `AbsSquare` and `SignPattern`. Each is reproduced verbatim from the source file, followed by the result it supports and the axiom list printed by `#print axioms`. `hunterFs s d₁ d₂ d₃` and `hunterF d₁ d₂ d₃` are F_s and F of Definition 2.1; `sumS₁`, `printedBracket`, `correctedBracket`, `printedD3one`, `correctedD3one`, `sumT2ord`, `sumR2un`, `sumT2cyc`, `sumR2cyc` and `sumS₃` are S_1, B_{pr}, B_{c}, D_{pr}, D_{c}, T_{ord}, R_{un}, T_{cyc}, R_{cyc} and S_3 of Section 4; `gnum q x` is $g_q(x)$ with real powers, `A m q` is $A(m, q)$, `sgn` is the sign function, `changes` counts sign changes of a list, and `gcoeffsHigh q`, `gcoeffsLow q` are the two ordered coefficient lists of Section 6. All definitions are over an arbitrary commutative ring except `gnum`, `A`, `sgn` and the coefficient lists, which are over $\mathbb{R}$.

Module `H4Lattice`**.**

```
theorem hunterF_nonneg (d₁ d₂ d₃ : ℕ) : 0 ≤ hunterF ((d₁ : ℕ) : ℤ) ((d₂ : ℕ) : ℤ) ((d₃ : ℕ) : ℤ)
```

Proposition 3.3; axioms: `propext`, `Quot.sound`.

```
theorem hunterF_d3_zero {R : Type*} [CommRing R] (d₁ d₂ : R) :
    hunterF d₁ d₂ 0 = (d₁ - d₂) ^ 2 * ((d₁ + d₂) ^ 2 + 4 * (d₁ + d₂) - 4)
```

Proposition 3.4; axioms: `propext`, `Quot.sound`.

```
theorem hunterF_half_zero_zero : hunterF ((1 : ℝ) / 2) 0 0 = -(7 / 16)
```

Proposition 3.4; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_not_nonneg_real :
    ¬ ∀ d₁ d₂ d₃ : ℝ, 0 ≤ d₁ → 0 ≤ d₂ → 0 ≤ d₃ → 0 ≤ hunterF d₁ d₂ d₃
```

Proposition 3.4; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem correctedBracket_eq {R : Type*} [CommRing R] (d₁ d₂ d₃ : R) :
    correctedBracket d₁ d₂ d₃ = sumS₁ d₁ d₂ d₃
```

Theorem 4.1; axioms: `propext`, `Quot.sound`.

```
theorem printedBracket_neg :
    printedBracket (5 : ℤ) 4 0 = -15 ∧ printedBracket (20 : ℤ) 19 2 = -2117
      ∧ sumS₁ (20 : ℤ) 19 2 = 6111
```

Theorem 4.1; axioms: `propext`, `Quot.sound`.

```
theorem sumS₁_nonneg (d₁ d₂ d₃ : ℝ) (h₃ : 0 ≤ d₃) (h₂ : d₃ ≤ d₂) (h₁ : d₂ ≤ d₁) :
    0 ≤ sumS₁ d₁ d₂ d₃
```

Theorem 4.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem printedD3one_off {R : Type*} [CommRing R] (d₁ d₂ : R) :
    hunterF d₁ d₂ 1 - printedD3one d₁ d₂ = (d₁ - d₂) ^ 2 * (d₁ + d₂) - 1
```

Theorem 4.2; axioms: `propext`, `Quot.sound`.

```
theorem printedD3one_witness :
    hunterF (2 : ℤ) 1 1 = 64 ∧ printedD3one (2 : ℤ) 1 = 62
```

Theorem 4.2; axioms: `propext`.

```
theorem correctedD3one_eq {R : Type*} [CommRing R] (d₁ d₂ : R) :
    hunterF d₁ d₂ 1 = correctedD3one d₁ d₂
```

Theorem 4.2; axioms: `propext`, `Quot.sound`.

```
theorem sumT2ord_eq_sumR2un {R : Type*} [CommRing R] (d₁ d₂ d₃ : R) :
    sumT2ord d₁ d₂ d₃ = sumR2un d₁ d₂ d₃
```

Proposition 4.3; axioms: `propext`, `Quot.sound`.

```
theorem sumT2cyc_ne_sumR2cyc :
    sumT2cyc (3 : ℤ) 2 2 = 2 ∧ sumR2cyc (3 : ℤ) 2 2 = 4
      ∧ sumT2cyc (4 : ℤ) 3 2 = -4 ∧ sumR2cyc (4 : ℤ) 3 2 = 26
```

Proposition 4.3; axioms: `propext`, `Quot.sound`.

```
theorem hunterF_decomposition {R : Type*} [CommRing R] (d₁ d₂ d₃ : R) :
    hunterF d₁ d₂ d₃ = sumS₁ d₁ d₂ d₃ + sumT2ord d₁ d₂ d₃ + sumS₃ d₁ d₂ d₃
```

Proposition 4.3; axioms: `propext`, `Quot.sound`.

```
theorem lc_identity {R : Type*} [CommRing R] (d₁ d₂ d₃ : R) :
    (d₂ + 2) ^ 2 * (d₂ + d₃ - d₁) + (d₁ + 2) ^ 2 * (d₁ + d₃ - d₂)
      = (d₁ - d₂) ^ 2 * (d₁ + d₂ + 4) + d₃ * ((d₂ + 2) ^ 2 + (d₁ + 2) ^ 2)
```

Proposition 3.7; axioms: `propext`, `Quot.sound`.

```
theorem lc_nonneg (d₁ d₂ d₃ : ℝ) (h₁ : 0 ≤ d₁) (h₂ : 0 ≤ d₂) (h₃ : 0 ≤ d₃) :
    0 ≤ (d₂ + 2) ^ 2 * (d₂ + d₃ - d₁) + (d₁ + 2) ^ 2 * (d₁ + d₃ - d₂)
```

Proposition 3.7; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_control_sharp :
    hunterF (0 : ℤ) 0 0 = 0 ∧ hunterF (1 : ℤ) 1 0 = 0 ∧ hunterF (1 : ℤ) 0 0 = 1
      ∧ hunterF (2 : ℤ) 1 1 = 64
```

Proposition 3.6; axioms: `propext`, `Quot.sound`.

```
theorem hunterF_control_not_pos : ¬ ∀ d₁ d₂ d₃ : ℕ, 1 ≤ hunterF ((d₁ : ℕ) : ℤ) d₂ d₃
```

Proposition 3.6; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterFs_three_neg : hunterFs (3 : ℤ) 1 0 0 = -2
```

Proposition 3.6; axioms: `propext`, `Quot.sound`.

```
theorem hunterFs_three_not_nonneg :
    ¬ ∀ d₁ d₂ d₃ : ℕ, 0 ≤ hunterFs (3 : ℤ) ((d₁ : ℕ) : ℤ) d₂ d₃
```

Proposition 3.6; axioms: `propext`, `Classical.choice`, `Quot.sound`.

Module `RealThreshold`.

```
theorem threshold_eq_sqrt : (2 * Real.sqrt 2 - 2) ^ 2 + 4 * (2 * Real.sqrt 2 - 2) = 4
```

Theorem 3.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem threshold_iff (s : ℝ) (hs : 0 ≤ s) : 4 ≤ s ^ 2 + 4 * s ↔ 2 * Real.sqrt 2 - 2 ≤ s
```

Theorem 3.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_split (x y z : ℝ) :
    hunterF (x + y + z) (y + z) z
      = x ^ 2 * ((x + 2 * y + 3 * z) ^ 2 + 4 * (x + 2 * y + 3 * z) - 4)
        + z * (2 * x * (4 - 4 * x - x ^ 2) + z * (12 + 12 * x - 5 * x ^ 2)
              + y * (16 + 16 * x - 2 * x ^ 2)
              + (3 * z ^ 3 + 8 * y * z ^ 2 + 10 * y ^ 2 * z + 4 * y ^ 3 + 4 * x * z ^ 2
                 + 10 * x * y * z + 6 * x * y ^ 2 + 12 * z ^ 2 + 24 * y * z + 16 * y ^ 2))
```

Theorem 3.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_split_high (x y z : ℝ) :
    hunterF (x + y + z) (y + z) z
      = x ^ 2 * (x ^ 2 + 4 * x - 4)
        + (x ^ 2 * (4 * z ^ 2 + 10 * y * z + 4 * y ^ 2 + 4 * z + 8 * y + 4 * x * z + 4 * x * y)
           + (3 * z ^ 4 + 8 * y * z ^ 3 + 10 * y ^ 2 * z ^ 2 + 4 * y ^ 3 * z + 4 * x * z ^ 3
              + 10 * x * y * z ^ 2 + 6 * x * y ^ 2 * z + 12 * z ^ 3 + 24 * y * z ^ 2
              + 16 * y ^ 2 * z + 12 * x * z ^ 2 + 16 * x * y * z + 12 * z ^ 2 + 16 * y * z
              + 8 * x * z))
```

Theorem 3.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_nonneg_real (d₁ d₂ d₃ : ℝ) (h₁ : 0 ≤ d₁) (h₂ : 0 ≤ d₂) (h₃ : 0 ≤ d₃)
    (hs : 4 ≤ (d₁ + d₂ + d₃) ^ 2 + 4 * (d₁ + d₂ + d₃)) :
    0 ≤ hunterF d₁ d₂ d₃
```

Theorem 3.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_threshold_sharp (t : ℝ) (ht : 0 < t) (ht4 : t ^ 2 + 4 * t < 4) :
    hunterF t 0 0 < 0
```

Theorem 3.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem hunterF_nonneg_real_of_one_le (d₁ d₂ d₃ : ℝ) (h₁ : 0 ≤ d₁) (h₂ : 0 ≤ d₂) (h₃ : 0 ≤ d₃)
    (h : 1 ≤ d₁ + d₂ + d₃) : 0 ≤ hunterF d₁ d₂ d₃
```

Corollary 3.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem threshold_control_needed :
    ((1 : ℝ) / 2) ^ 2 + 4 * ((1 : ℝ) / 2) = 9 / 4 ∧ hunterF ((1 : ℝ) / 2) 0 0 = -(7 / 16)
```

Proposition 3.5; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem threshold_control_not_pos : hunterF (1 : ℝ) 1 0 = 0 ∧ (4 : ℝ) ≤ (1 + 1 + 0) ^ 2 + 4 * (1 + 1 + 0)
```

Proposition 3.5; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem threshold_control_sharp_side :
    (0 : ℝ) < hunterF ((9 : ℝ) / 10) 0 0 ∧ hunterF ((8 : ℝ) / 10) 0 0 < 0
```

Proposition 3.5; axioms: `propext`, `Classical.choice`, `Quot.sound`.

Module `AbSquare`.

```
theorem weighted_two_largest_ge (γ₁ γ₂ γ₃ u v w : ℝ)
    (hγ₁ : 0 < γ₁) (hγ₂ : 0 < γ₂) (hγ₃ : 0 < γ₃)
    (hvu : v ≤ u) (hwv : w ≤ v) (hw : 0 ≤ w)
    (hdom : γ₁ ≤ γ₂ + γ₃)
    (hnorm : γ₁ * u + γ₂ * v + γ₃ * w = 1) :
    2 / (γ₁ + γ₂ + γ₃) ≤ u + v
```

Proposition 5.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem abc_sq_ge_two_div_n (γ₁ γ₂ γ₃ n : ℕ) (a b c : ℝ)
    (hγ₁ : 1 ≤ γ₁) (hγ₂ : 1 ≤ γ₂) (hγ₃ : 1 ≤ γ₃) (hn : γ₁ + γ₂ + γ₃ = n)
    (hab : b ^ 2 ≤ a ^ 2) (hbc : c ^ 2 ≤ b ^ 2)
    (hdom : γ₁ ≤ γ₂ + γ₃)
    (hnorm : (γ₁ : ℝ) * a ^ 2 + (γ₂ : ℝ) * b ^ 2 + (γ₃ : ℝ) * c ^ 2 = 1) :
    2 / (n : ℝ) ≤ a ^ 2 + b ^ 2
```

Proposition 5.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem weights_control_sharp :
    (1 : ℝ) * (1 / 3) + (1 : ℝ) * (1 / 3) + (1 : ℝ) * (1 / 3) = 1
      ∧ (2 : ℝ) / (1 + 1 + 1) = 1 / 3 + 1 / 3
```

Proposition 5.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem weights_control_dominant :
    ¬ ∀ γ₁ γ₂ γ₃ u v w : ℝ, 0 < γ₁ → 0 < γ₂ → 0 < γ₃ → v ≤ u → w ≤ v → 0 ≤ w →
      γ₁ * u + γ₂ * v + γ₃ * w = 1 → 2 / (γ₁ + γ₂ + γ₃) ≤ u + v
```

Proposition 5.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem weights_control_ordering :
    ¬ ∀ γ₁ γ₂ γ₃ u v w : ℝ, 0 < γ₁ → 0 < γ₂ → 0 < γ₃ → v ≤ u → 0 ≤ w → γ₁ ≤ γ₂ + γ₃ →
      γ₁ * u + γ₂ * v + γ₃ * w = 1 → 2 / (γ₁ + γ₂ + γ₃) ≤ u + v
```

Proposition 5.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem weights_control_best :
    ¬ ∀ γ₁ γ₂ γ₃ u v w : ℝ, 0 < γ₁ → 0 < γ₂ → 0 < γ₃ → v ≤ u → w ≤ v → 0 ≤ w → γ₁ ≤ γ₂ + γ₃ →
      γ₁ * u + γ₂ * v + γ₃ * w = 1 → 2 / (γ₁ + γ₂ + γ₃) < u + v
```

Proposition 5.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

Module `SignPattern`**.**

```
theorem gnum_at_one (q : ℝ) : gnum q 1 = 0
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_exp (q t : ℝ) :
    gnum q (Real.exp t) =
      -(Real.exp ((q + 2) * t)) + q * Real.exp ((q + 1) * t) - (q + 1) * Real.exp (q * t)
        + (q + 1) * Real.exp (2 * t) - q * Real.exp t + 1
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem A_zero (q : ℝ) : A 0 q = -1
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem A_one (q : ℝ) : A 1 q = 0
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem A_two (q : ℝ) : A 2 q = 0
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem A_three (q : ℝ) : A 3 q = q * (q + 1) * (q - 4)
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem A_three_pos_iff (q : ℝ) (hq : 0 < q) : 0 < A 3 q ↔ 4 < q
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_exponent_order_high (q : ℝ) (hq : 2 < q) :
    (0 : ℝ) < 1 ∧ (1 : ℝ) < 2 ∧ (2 : ℝ) < q ∧ q < q + 1 ∧ q + 1 < q + 2
```

Proposition 6.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_exponent_order_low (q : ℝ) (hq₁ : 1 < q) (hq₂ : q < 2) :
    (0 : ℝ) < 1 ∧ (1 : ℝ) < q ∧ q < 2 ∧ (2 : ℝ) < q + 1 ∧ q + 1 < q + 2
```

Proposition 6.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_sgn_high (q : ℝ) (hq : 2 < q) :
    (gcoeffsHigh q).map sgn = [1, -1, 1, -1, 1, -1] ∧ changes [1, -1, 1, -1, 1, -1] = 5
```

Proposition 6.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_sgn_low (q : ℝ) (hq₁ : 1 < q) (hq₂ : q < 2) :
    (gcoeffsLow q).map sgn = [1, -1, -1, 1, 1, -1] ∧ changes [1, -1, -1, 1, 1, -1] = 3
```

Proposition 6.2; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_sgn_control : changes [1, -1, 1, -1, 1, -1] ≠ changes [1, -1, -1, 1, 1, -1]
```

Proposition 6.2; axioms: `propext`.

```
theorem A_three_control : A 3 (5 : ℝ) = 30 ∧ A 3 (2 : ℝ) = -12
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

```
theorem gnum_control_nonzero : gnum 5 (2 : ℝ) = 15
```

Proposition 6.1; axioms: `propext`, `Classical.choice`, `Quot.sound`.

References

[1] S. Brazitikos and C. Pandis, *Sharp inequalities for symmetric polynomials, Hunter's conjecture, and moments of exponential random variables*, arXiv:2512.12254v1 [math.PR], 13 December 2025. As of 3 September 2026 this is the only version. Statement numbers are those of the typeset v1 (the arXiv PDF and a build of the e-print agree): Proposition 3.1 is the h_4 statement, Theorem 3.2 is Hunter's conjecture, Proposition 3.3 the Lagrange reduction, Definition 4.4 defines ρ, Theorem 4.5 is the integer-exponent inequality, Conjecture 4.8 (the source's label is `Conj1`) the real-exponent conjecture, Proposition 4.12 its case $n = 2$, and Lemma 4.13 the log-concavity lemma.

[2] S. Brazitikos, C. Tang and T. Tkocz, *Moments of sums of exponentials, beyond CHS*, arXiv:2602.03058v1 [math.PR], 3 February 2026; the only version as of 3 September 2026. Its theorems are numbered globally: Theorem 1 is the L_p bound for $p \geq 2$, Remark 2 states the fixed-n real-exponent form of Hunter's conjecture, Theorem 5 is the Schur-monotonicity theorem quoted in Section 1, Lemmas 6 and 7 are the integration-by-parts and difference identities, and (8) is the critical-point identity used in Remark 7.2.

[3] D. B. Hunter, *The positive-definiteness of the complete symmetric functions of even order*, Math. Proc. Cambridge Philos. Soc. **82** (1977), no. 2, 255–258; `doi:10.1017/S030500410005386X`.

[4] D. R. Curtiss, *Recent extensions of Descartes' rule of signs*, Ann. of Math. (2) **19** (1918), no. 4, 251–278; `doi:10.2307/1967494`. Cited by [1] for the rule of signs with real exponents.

[5] S. Brazitikos, A. Giannopoulos, P. Valettas and B.-H. Vritsiou, *Geometry of Isotropic Convex Bodies*, Mathematical Surveys and Monographs **196**, American Mathematical Society, Providence, RI, 2014; `doi:10.1090/surv/196`. The reference [1] gives for its Lemma 4.13.

[6] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: A. Platzer and G. Sutcliffe (eds.), Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, Cham, 2021, pp. 625–635; `doi:10.1007/978-3-030-79876-5_37`.

[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; `doi:10.1145/3372885.3373824`.