THE SMALLEST POLYOMINO NETS THAT FOLD INTO THEIR BOX IN SEVERAL ESSENTIALLY DIFFERENT WAYS

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Abstract. A polyomino that folds onto the surface of a box may do so in more than one essentially different way. We make the count precise – the rotation group $\mathrm{Rot}(B)$ acts freely on the set of foldings of a net P into B, so $|\mathrm{Rot}(B)|$ divides that set's cardinality and the number of orbits $\mathrm{ways}(B, P)$ is well defined – and then determine the smallest surface areas at which multiplicity occurs. That 10 is the smallest area at which multiple folding ways exist is asserted, without proof and without a definition of "different", by Uehara; we prove it, together with the statement that 10 is also the smallest area at which some net folds three ways, and the exclusion below area 10 uses no enumeration of polyominoes at all. We then exhibit a 16-cell net folding into $1 \times 2 \times 2$ in four ways and three 18-cell nets folding into $1 \times 1 \times 4$ in four ways – the smallest four-way net previously exhibited has area 28 – and prove that no polyomino of surface area below 14 folds into its box in four essentially different ways, so the smallest such area is 14 or 16. That bound exhausts the cube and $1 \times 1 \times 2$ over enumerations of their developments whose completeness is itself proved rather than programmed. All statements are machine-checked in Lean 4.

1. Introduction

The objects. A *polyomino* is a finite, nonempty, edge-connected set of unit cells of $\mathbb{Z}^2$. An integer *box* $B = a \times b \times c$ has surface area $s(B) = 2(ab + ac + bc)$, measured in unit squares. The polyomino P *folds into* B when it can be wrapped onto the surface of B, covering every surface square exactly once, with all creases along the grid lines of the cells; such a P is a *net* of B. A polyomino that is a net of several boxes at once is a *common unfolding* of them.

Almost all of the literature on this model counts *nets*. Following Qian, Wang, Subercaseaux and Heule [7], write $\mathcal{B}(s)$ (their $P(s)$) for the set of normalised triples $a \leq b \leq c$ with $2(ab + ac + bc) = s$, and $\Psi(k)$ for the least s such that some k-element subset of $\mathcal{B}(s)$ has a common unfolding. The classical results are of this shape: eleven of the thirty-five free hexominoes are nets of the cube; $\Psi(2) = 22$, with 2263 common unfoldings of $1 \times 1 \times 5$ and $1 \times 2 \times 3$ [1]; $\Psi(3) \leq 532$ [8], improved to $\Psi(3) \leq 106$ by Demaine et al. [2]; and $\Psi(3) > 58$ [7], which refutes the conjecture $\Psi(3) = 46$ that [7] attributes to Xu, Horiyama, Shirakawa and Uehara [12]. Counting the nets of a fixed box is harder than it looks: [7] reports that even the number of nets of $1 \times 1 \times 2$ was unknown before their SAT search returned 723.

A different count. This paper is about a question one level down. Fix a net P and a single box B. How many ways are there to fold P onto B?

Turning the folded box in space carries a folding to another folding of the same net, so the raw number of foldings is never the interesting quantity; the interesting one is the number of orbits under the rotation group of the box. Call this ways(B, P). Four sources bear on it. Mizunashi, Horiyama and Uehara [6] give a pseudo-polynomial algorithm computing all ways of folding a given polygon to boxes, applied there to nine polyominoes; they publish no census over the nets of a fixed box and no minimal-area statement. Tadaki and Amano [9] search by SAT for nets with ways ≥ 2, counting the ways with that algorithm, and report as their headline that "we found thousands of such developments including a polyomino of area 52 that can fold into a box of size $1 \times 2 \times 8$ in five different ways". Their search is a sampling search and they say so, noting that a badly chosen parameter makes them "fail to find a development although it actually exists"; the smallest surface area their search runs at is 28, although their introduction already displays an area-22 net that folds into $1 \times 2 \times 3$ in two ways; and they state no minimality result. Uehara's book [11, §3.3.5, p. 39] exhibits a 10-cell net that folds into a box "of the same type in three different ways" and asserts that it "is the net with the smallest area where multiple folding ways exist", sketching in one sentence the argument we make precise in Theorem 14; no proof, no census and no definition of "different" accompany it.

So the literature offers fold counts for individual nets and one asserted minimal area, but no census over the nets of any box, no bound of any kind on ways at any area between 12 and 26, and nothing at all about four foldings of one net into one box. That is the gap this paper closes at the bottom end. What was and was not searched is set out in §10, and the reader should weigh it before reading any claim below as a priority claim.

What is settled here. Throughout, $\mathrm{Fold}(B, P)$ is the set of foldings of P into B and $\mathrm{Rot}(B)$ the rotation group of B; both are defined combinatorially in Section 2. Section 3 shows that $\mathrm{Rot}(B)$ acts freely on $\mathrm{Fold}(B, P)$, so that

$$\mathrm{ways}(B, P) \;=\; \frac{|\,\mathrm{Fold}(B, P)|}{|\,\mathrm{Rot}(B)|}$$

is an orbit count. With that in place:

- **Theorem 14.** 10 is the smallest surface area at which a polyomino folds into its box in two essentially different ways, and also the smallest at which one folds in three; below area 10 every net of every box folds in exactly one way. The conclusion is asserted without proof in [11, p. 39]; what is new here is the proof and the definition it is a proof about.
- **Theorem 17.** A 16-cell polyomino folds into $1 \times 2 \times 2$ in four essentially different ways, and three 18-cell ones fold into $1 \times 1 \times 4$ in four. The smallest four-way net exhibited previously is Tadaki–Amano's, of area 28.
- **Theorem 19.** No polyomino of surface area below 14 folds into its box in four essentially different ways; hence the smallest surface area admitting a four-way net is 14 or 16.

The proof of Theorem 14 needs no enumeration of polyominoes whatsoever: areas 8 and 12 carry no box at all, so below area 10 the only box is the cube, and there the frame count and the group order coincide – the cube has 24 frames on its surface and $|\,\mathrm{Rot}\,| = 24$ – so one orbit already exhausts the frames. Theorem 19, by contrast, is genuinely exhaustive: it runs two boxes to completion over enumerations of their developments whose completeness is itself a theorem (Section 5) rather than a property of a program. Section 6.4 says exactly which finite searches were run. The remaining case, area 14, is a matter of compute rather than of mathematics (Remark 21).

Method, and two limitations. Everything here is stated against a combinatorial definition of folding (Section 2) and checked in Lean 4 (Section 9). The definition is combinatorial by choice: we do *not* formalise that a folding in this sense is realised by a continuous folding motion in three-space. That is precisely why [7] declines to define folding at all, writing that "a precise mathematical definition of folding/unfolding turns out to be pretty intricate, using the entirety of Chapter 11 in the book of Demaine and O'Rourke" [3], and that they will "mostly stick to intuition". Nothing below depends on the missing half; what the definition does capture is checked against published data at six independent points (Section 8). Section 10 records the limits of the literature search behind the novelty claims, and exactly how much of [11] we were able to read.

We also make no attempt at the two questions with which [2] closes – the smallest area of a polyomino covering three cuboids, and whether one covers four – both being searches over ranges where a single published instance has cost days of computation. What Section 8 contributes to the first is arithmetic: given $\Psi(3) > 58$ from [7] and the area-106 net of [2], exactly ten areas remain to be searched, carrying 37 boxes.

2. Preliminaries

2.1. **Boxes, squares and frames.** Place the box $B = a \times b \times c$ as $[0, 2a] \times [0, 2b] \times [0, 2c]$ in *doubled* integer coordinates. Then the centre of every surface unit square is a point of $\mathbb{Z}^3$ with exactly one even coordinate, that coordinate being 0 or the full extent along its axis; which coordinate is even names the face. Write $Q(B) \subset \mathbb{Z}^3$ for the set of these centres, so $|Q(B)| = s(B)$, and $n(p)$ for the outward unit normal of the face carrying $p \in Q(B)$. No halves, no rationals: the whole model lives in $\mathbb{Z}^3$.

A *frame* on the surface is a pair $\sigma = (p, u)$ with $p \in Q(B)$ and $u \in \mathbb{Z}^3$ an axis unit vector orthogonal to $n(p)$: the image of the net's own "up" direction at one cell. The implied "right" direction is $u \times n(p)$. Write $S(B)$ for the set of frames, so $|S(B)| = 4\,s(B)$.

Rolling one square in one of the four plane directions $d \in \{0, 1, 2, 3\}$ (up, right, down, left) is the map $\mathrm{step}_d \colon S(B) \to S(B)$ defined as follows. Put $n = n(p)$, let w be u, $u \times n$, $-u$ or $-(u \times n)$ according to d, and let $e = p + w$ be the midpoint of the shared edge. If the coordinate of e along the axis of w is 0 or the full extent – exactly the condition that the step leaves the current face – then, with $m = n \times w$,

$$\mathrm{step}_d(p, u) \;=\; \big(\, p + w - n, \;\; \langle u, m\rangle\, m - \langle u, w\rangle\, n \,\big),$$

a right-angle rotation of the frame about the box edge; otherwise the step stays on the face and $\mathrm{step}_d(p, u) = (p + 2w,\ u)$.

2.2. **Polyominoes and foldings.** Cells of $\mathbb{Z}^2$ are adjacent when they differ by a unit vector; a *polyomino* is a finite nonempty edge-connected set of cells. For a cell c and a direction d we write $c + e_d$ for the neighbour of c in direction d, matching the indexing of step_d.

Definition 1 (folding)**.** Let B be a box and P a polyomino. A map $\varphi\colon P \to S(B)$ is a *folding of P into B* when

(L) *locality:* whenever $c \in P$ and $c{+}e_d \in P$, we have $\varphi(c{+}e_d) = \mathrm{step}_d(\varphi(c))$;
(I) *injectivity:* the induced map $P \to Q(B)$, $c \mapsto$ the square of $\varphi(c)$, is injective;
(S) *surjectivity:* that induced map is onto $Q(B)$.

We write $\mathrm{Fold}(B, P)$ for the set of foldings, and say that P *folds into* B when this set is nonempty.

Clause (L) is the isometry condition and is the whole content of the word "folding". Imposed at *every* plane adjacency rather than only along a spanning tree, it excludes what [7] calls *touching* solutions – those need the paper cut along an edge, as that paper itself observes – while (I) and (S) exclude its *overlapping* solutions. So Definition 1 is intended to capture exactly the *simple* nets of [7], and Section 8 reports the evidence that it does.

Two elementary consequences are worth isolating, because they are the arithmetic half of the negative controls in Section 7.

Proposition 2. *If P folds into B then $|P| = s(B)$. Consequently two boxes with a common unfolding have the same surface area.*

Proof. By (I) the images of the cells of P are $|P|$ distinct elements of $Q(B)$, and by (S) they exhaust $Q(B)$; now $|Q(B)| = s(B)$. □

2.3. **Listings, and deciding whether a net folds.** A folding is rigid: by (L), fixing the frame at one cell forces the frame at every cell reachable from it, hence – P being connected – at every cell. This is what makes folding decidable. Call a listing $c_1, \dots, c_k$ of a polyomino *chained* when each c_i with $i \geq 2$ is adjacent to some earlier c_j. Given a chained listing and a frame σ at c_1, one pass from left to right propagates σ to a table of frames, and one then checks the table against every clause of Definition 1. Writing $\mathrm{folds}(B, P)$ for "some frame passes", we have:

Proposition 3 (soundness and completeness of the decision procedure)**.** *For a polyomino P presented in a chained listing,* $\mathrm{folds}(B, P) =$ *true if and only if P folds into B.*

Completeness – the "only if" – is what makes "this polyomino does *not* fold into that box" a theorem rather than a failed search, and every negative statement below depends on it. Its proof is that the propagation pass reproduces *any* folding agreeing with it at the first cell. Section 4 removes the chained-listing hypothesis.

Finally we write $\mathcal{B}(s)$ for the set of boxes $1 \leq a \leq b \leq c$ of surface area s; it is finite, since $s(B) \geq 2c$ bounds every side, and membership in it is decidable.

Proposition 4. *$B \in \mathcal{B}(s)$ if and only if $1 \leq a \leq b \leq c$ and $s(B) = s$. In particular $\mathcal{B}(6) = \{1{\times}1{\times}1\}$, $\mathcal{B}(10) = \{1{\times}1{\times}2\}$, and $\mathcal{B}(s) = \emptyset$ for $s \in \{7, 8, 9, 11, 12, 13\}$.*

The emptiness statements are finite checks; they are what makes the surface-area sequence $6, 10, 14, 16, 18, \dots$ start with two isolated values, and they are used in both extremal theorems below.

3. Counting foldings, and the rotation group

3.1. **The count exists.** Definition 1 constrains φ only on P, so the foldings of P are not literally a finite set of functions $\mathbb{Z}^2 \to S(B)$; what is finite is the set of restrictions to P, and that is what $\mathrm{Fold}(B,P)$ means throughout. Fix a chained listing of P with first cell c_1. The map

$$\mathrm{Fold}(B,P) \longrightarrow S(B), \qquad \varphi \longmapsto \varphi(c_1)$$

is injective, by rigidity, and its image is exactly the set of frames the decision procedure of Proposition 3 accepts. So $\mathrm{Fold}(B,P)$ is finite, computable, and of cardinality at most $|S(B)| = 4\,s(B)$.

3.2. **Box symmetries transport foldings.** Let g be an affine map of $\mathbb{Z}^3$ which permutes $S(B)$, commutes with step_d for every d, and restricts to a bijection of $Q(B)$. Call such a g a *symmetry of B*.

Proposition 5 (transport)**.** *If g is a symmetry of B and $\varphi \in \mathrm{Fold}(B,P)$, then $g \circ \varphi \in \mathrm{Fold}(B,P)$.*

Proof. Clause by clause. (L) is the commutation of g with step_d; (I) is injectivity of g on $Q(B)$ composed with injectivity of φ; (S) is surjectivity of g on $Q(B)$ composed with that of φ. No computation enters. □

Let $\mathrm{Rot}(B)$ be the set of maps obtained from the 24 rotations of the cubic lattice by translating each so that the centre of B is fixed, and keeping those that carry the surface of B into itself. This is a construction, not a table: $|\mathrm{Rot}(B)|$ comes out as 24, 8 or 4 according as three, two or no side lengths of B agree.

We use the following seven finite properties of a box B, which we abbreviate as (G): the frames $S(B)$ are pairwise distinct; $\mathrm{Rot}(B)$ is duplicate-free, contains the identity, is closed under composition and contains an inverse for each of its elements; every element of $\mathrm{Rot}(B)$ is a symmetry of B; and $\mathrm{Rot}(B)$ acts *freely* on $S(B)$, in the strong sense that $g\cdot\sigma = h\cdot\sigma$ for a single $\sigma \in S(B)$ already forces $g = h$. Property (G) is decidable and is verified once per box for every box occurring in this paper.

Theorem 6 (divisibility)**.** *Let B satisfy* (G) *and let P be any polyomino. Then* $|\mathrm{Rot}(B)|$ *divides* $|\mathrm{Fold}(B,P)|$.

Proof. Identify $\mathrm{Fold}(B,P)$ with its image in $S(B)$ under $\varphi \mapsto \varphi(c_1)$. By Proposition 5 that image is stable under $\mathrm{Rot}(B)$, and the action is free because it is already free on $S(B)$. A finite group acting freely on a finite set partitions it into orbits of size $|\mathrm{Rot}(B)|$. (The Lean proof is a direct induction: peel off one orbit, whose $|\mathrm{Rot}(B)|$ elements are distinct by freeness, note that the complement is still stable – this is where inverses are used – and recurse. No orbit representatives are chosen and no quotient is formed.) □

Definition 7. For B satisfying (G) and P a polyomino, the number of *essentially different* foldings is $\mathrm{ways}(B,P) := |\mathrm{Fold}(B,P)|/|\mathrm{Rot}(B)|$, the number of $\mathrm{Rot}(B)$-orbits.

Note that ways quotients by turning the *box*, not by symmetries of the net in the plane; two foldings related by a symmetry of P that does not come from a rotation of B are counted separately. This is the reading under which Tadaki–Amano's six published counts come out right (Section 8).

4. Re-rooting a polyomino: a folklore lemma

The decision procedure of Proposition 3 wants its polyomino in a chained listing, which is free when the net is data one writes down and awkward when the net is produced by a proof: an enumeration hands over a *set* of cells, and a net about to be re-rooted at the preimage of a prescribed surface square has no reason to be listed starting there. The following removes the hypothesis.

Lemma 8. *Let P be a polyomino and $r \in P$. Then P admits a chained listing whose first cell is r.*

Proof. Grow greedily from $\{r\}$, at each step appending some cell of P adjacent to what has been taken. The growth cannot stall: in a connected set, a nonempty proper subset always has a cell of the complement adjacent to it – walk a connectivity path from inside the subset to a cell outside and take the first step that leaves. A length count then forces the growth to exhaust P. □

We stress that Lemma 8 is folklore, and is treated as such by every search in this area: the connectivity encoding of [9] is a breadth-first "ignition" from a single seed, [7] defines the orientation of a graph towards an arbitrary vertex, and [2] uses the statement outright as its completeness argument ("The idea of this algorithm is to search for nets by trying every possible way in which a breadth-first search algorithm can explore a net in the shape of a polyomino. If we do that, we will have searched all non-overlapping nets."). It is stated here as a lemma because the arguments of Section 5 need it in a form that can be applied, not because it is new.

Three consequences follow, all of them bookkeeping:

Proposition 9. *Let P be a polyomino. Then*

(i) *deciding whether P folds into B needs no chained listing: applying the procedure of Proposition 3 to the re-rooted listing of Lemma 8 decides* $\mathrm{Fold}(B, P) \neq \emptyset$ *for P in* any *listing;*
(ii) $|\mathrm{Fold}(B, P)|$ *depends only on the set of cells, not on the listing;*
(iii) $|\mathrm{Fold}(B, P)|$ *is unchanged by translating P.*

Proof. For (ii), send a frame accepted at the first cell of one listing to the folding it determines, read at the first cell of the other; this is injective in both directions, and comparing lengths gives equality. (iii) is the same characterisation together with the observation that translating the net conjugates the folding. (i) is Lemma 8 plus the remark that Definition 1 sees P only through membership. □

Parts (ii) and (iii) turn "no net of this box folds four ways" into a statement about nets rather than listings, and are what the next section consumes.

5. Enumerating the developments of a box, completely

To bound ways over *all* nets of a box one needs a list of nets that provably misses none. We build one from the box side.

By a *development* of B rooted at a frame σ_0 we mean a partial assignment: a finite edge-connected set of cells containing the origin, together with a map from it to $S(B)$ satisfying clauses (L) and (I) of Definition 1 and sending the origin to σ_0. A folding is a development whose cell set has $s(B)$ cells.

The enumeration $\mathcal{G}_n(B, \sigma_0)$ is built level by level: level 0 is the single origin cell carrying σ_0; level $n+1$ extends each entry of level n by one neighbouring cell whose frame is *forced* by the already placed neighbours, accepting the extension exactly when all placed neighbours force the same frame and the surface square it lands on is not already used, and then merges entries with equal cell sets. A node stores the developed table itself, so an extension costs $O(k)$ rather than a fresh $O(k^2)$ propagation.

Theorem 10 (completeness of the enumeration)**.** *Every set of $n+1$ cells carrying a development of B rooted at σ_0 occurs, as a cell set, at level n of $\mathcal{G}(B, \sigma_0)$.*

Proof sketch. Induct on n. A chained listing with its last cell deleted is still chained and the deleted cell is adjacent to the rest, so the restriction of the development to the first n cells is again a development and, by induction, occurs at level $n-1$. That the acceptance test does not reject the last cell is the locality clause: the developed table of a chained partial net is literally the graph of the assignment, so the frames forced by the placed neighbours are the true ones and agree, and injectivity says the square is unused. The one point needing care is the level merge: it sorts by cell set and drops adjacent duplicates, and what the induction needs is only that *every key that occurs still occurs*, which holds because the sort is a permutation – no order-theoretic property of the comparison is used. □

The acceptance test is deliberately one-sided: it may be generous, admitting tables that extend to nothing, but it may not be strict.

Combining Theorem 10 with Proposition 9:

Corollary 11. *Assume the frames of B are pairwise distinct (the first clause of* (G)*). Let P be a polyomino, $\varphi \in \mathrm{Fold}(B, P)$, and $c_0 \in P$ with $\varphi(c_0) = \sigma_0$. Then some entry T of level $|P| - 1$ of $\mathcal{G}(B, \sigma_0)$ has $|\,\mathrm{Fold}(B, \text{cells of } T)| = |\,\mathrm{Fold}(B, P)|$.*

That is: up to translation and re-listing, every net one of whose foldings hits the frame σ_0 is one of the enumerated cell sets, *with the same fold count*.

5.1. **Removing the restriction on the frame.** Corollary 11 still asks a folding to hit the frame σ_0, not merely the square under it. Fix a surface square q_0 and a frame σ_0 over it. A folding covers every square, so it sends some cell to q_0, carrying one of the four frames there. If the rotations of B fixing q_0 act transitively on those four frames – a finite condition on the box alone, and one we check per box – then turning the folded box by such a rotation gives a folding of the *same* net whose frame at that cell is σ_0, by Proposition 5. For a box $1 \times b \times c$ and q_0 the centre of an end face, the stabiliser of q_0 is the group of rotations about the long axis, of order four, and the condition holds.

Proposition 12. *Let B satisfy* (G)*, let $q_0 \in Q(B)$, and let the rotations of B carry every frame over q_0 to σ_0. Then for every polyomino P and every $\varphi \in \mathrm{Fold}(B, P)$ there is $\psi \in \mathrm{Fold}(B, P)$ and a cell $c_0 \in P$ with $\psi(c_0) = \sigma_0$.*

5.2. **One frame per orbit.** Computing $|\,\mathrm{Fold}(B, P)|$ by testing all $4\,s(B)$ frames is wasteful, since by Proposition 5 the accepted set is a union of $\mathrm{Rot}(B)$-orbits. Choose representatives $R(B) \subseteq S(B)$, one per orbit, greedily, and check (again, once per box) that their orbits cover $S(B)$.

Proposition 13 (orbit bound)**.** *Let B satisfy* (G) *and let $R(B)$ cover $S(B)$. For every polyomino P,*

$$|\,\mathrm{Fold}(B, P)| \;\le\; \big|\{\rho \in R(B) : \rho \text{ is accepted for } P\}\big| \cdot |\,\mathrm{Rot}(B)|.$$

Proof. An accepted frame lies in the orbit of some representative; that representative is itself accepted, by applying the inverse rotation and Proposition 5; and an orbit has at most $|\operatorname{Rot}(B)|$ elements. $\square$

Only the inequality is proved, because only it is used: an exhaustive statement of the form "no net folds more than k ways" needs an upper bound on every net's fold count. For $1 \times 1 \times 2$ this replaces 40 frame tests per net by 5, and the reduction is what brings the computation of Theorem 19 inside budget: the direct 40-frame run was measured at roughly two CPU-hours and abandoned, against thirteen CPU-minutes for the orbit version.

6. The extremal results

6.1. **Two and three ways.**

Theorem 14. (i) *Let B be a box with $s(B) < 10$ and let P be a polyomino folding into B. Then* $\operatorname{ways}(B, P) = 1$.

(ii) *There are* 10*-cell polyominoes W_2 and W_3 with* $|\operatorname{Fold}(1{\times}1{\times}2, W_2)| = 16$, $\operatorname{ways}(1{\times}1{\times}2, W_2) = 2$ *and* $|\operatorname{Fold}(1{\times}1{\times}2, W_3)| = 24$, $\operatorname{ways}(1{\times}1{\times}2, W_3) = 3$.

Hence 10 *is the smallest surface area at which a polyomino folds into its box in two essentially different ways, and it is also the smallest at which one folds in three.*

Proof. (i) By Proposition 4 the only surface area below 10 carrying a box is 6, and $\mathcal{B}(6) = \{1{\times}1{\times}1\}$. For the cube, $|S| = 4 \cdot 6 = 24$ and $|\operatorname{Rot}| = 24$. Since Fold injects into S we get $|\operatorname{Fold}| \leq 24$; since P folds, $|\operatorname{Fold}| \geq 1$; and by Theorem 6, 24 divides $|\operatorname{Fold}|$. Hence $|\operatorname{Fold}| = 24$ and ways $= 1$.

(ii) The two witnesses are

$$W_2 = \{(0,1),(1,1),(1,2),(2,1),(1,0),(1,3),(1,4),(2,3),(0,3),(1,5)\},$$
$$W_3 = \{(0,0),(0,1),(0,2),(1,1),(2,1),(3,1),(3,2),(4,1),(3,0),(5,1)\},$$

and the four numbers are computed by testing all 40 frames of $1 \times 1 \times 2$ against each. Since $|\operatorname{Rot}(1{\times}1{\times}2)| = 8$, the orbit counts are 2 and 3. $\square$

Part (i) uses no polyomino at all: no list of hexominoes, no enumeration, no completeness argument. The only facts about the box that enter are that $\mathcal{B}(s)$ is empty for $s \in \{7, 8, 9\}$, that $|S(1{\times}1{\times}1)| = 24$, that the cube satisfies (G) and that $|\operatorname{Rot}(1{\times}1{\times}1)| = 24$ – none of which mentions a net.

Remark 15 (attribution). The conclusion of Theorem 14 is not new. Uehara states it, without proof, in [11, §3.3.5, p. 39]. Having exhibited in his Fig. 3.9 a 10-cell net that folds into a box "of the same type in three different ways", he writes:

> It is well known that there are eleven different nets in a unit cube by edge-unfolding (Fig. 1.1), and each of them is folded in a unique manner. Therefore, Fig. 3.9 is the net with the smallest area where multiple folding ways exist.

That is the argument of part (i) in one sentence, with the cube case taken as known rather than derived; and because his Fig. 3.9 folds three ways, the same assertion covers the three-way half of the theorem as well. What Theorem 14 adds is a proof, and a definition for the proof to be about: "essentially different" in the sense of Definition 7, the divisibility that makes that definition well posed

(Theorem 6), the cube case deduced from those rather than quoted, and a machine-checked derivation. The book defines no notion of "different" for box foldings, proves nothing here, and gives no census.

His net is not ours. Read off the figure, Fig. 3.9 is the polyomino

$$\{(1,0),\ (0,1),(1,1),(2,1),\ (1,2),\ (1,3),\ (0,4),(1,4),(2,4),\ (1,5)\},$$

a column of six cells with a pair of side cells at the second and the fifth, which is not congruent to W_3. The census of Remark 31 finds four three-way nets of $1 \times 1 \times 2$ up to congruence, and W_3 and Uehara's are two different ones among them; that identification rests on the census, which is a computation and not a theorem.

Multiplicity is not a freak: of the 723 nets of $1 \times 1 \times 2$, the computation of Remark 31 finds 68 folding two ways and 4 folding three.

Remark 16. A published statement looks similar and is a different question: 22 is the smallest area carrying a common net of two *different* boxes [1, 7]. That is one net and two boxes; Theorem 14 is one net, one box and several foldings of it.

6.2. **Four ways: witnesses.**

Theorem 17. *Let*

$$\begin{aligned} W_4 = \{&(0,1),(1,1),(1,2),(2,1),(1,3),(2,2),(2,0),(3,2),\\ &(3,3),(4,3),(4,4),(5,3),(4,5),(5,4),(5,2),(6,4)\}. \end{aligned}$$

Then $|W_4| = 16 = s(1{\times}2{\times}2)$*,* $|\operatorname{Fold}(1{\times}2{\times}2, W_4)| = 32$ *and* $\operatorname{ways}(1{\times}2{\times}2, W_4) = 4$*. There are, further, three pairwise non-congruent* 18*-cell polyominoes each with* $|\operatorname{Fold}(1{\times}1{\times}4,\cdot)| = 32$ *and* $\operatorname{ways}(1{\times}1{\times}4,\cdot) = 4$*; a* 14*-cell polyomino with* $\operatorname{ways}(1{\times}1{\times}3,\cdot) = 3$*; and* 22*-cell polyominoes with* $\operatorname{ways}(1{\times}1{\times}5,\cdot) = 2$ *and* $= 3$*. The seven witnesses of area at most* 18 *are pairwise non-congruent under translation and the eight symmetries of the square, so they are seven shapes and not seven listings.*

Proof. Each fold count is obtained by testing all $4\,s(B)$ frames of the relevant box against the given cell list, and each is then divided by $|\operatorname{Rot}(B)| = 8$ through Theorem 6 rather than by arithmetic on the printed numbers. Non-congruence is decided by comparing canonical forms under the eight lattice symmetries. □

The smallest four-way net exhibited anywhere previously is Tadaki–Amano's, of area 28 [9, Fig. 7], found by SAT sampling; W_4 is smaller by 12. Theorem 17 claims nothing about minimality or uniqueness at area 16; that is the subject of the next theorem and of Remark 31.

Remark 18 (a published "four different ways" that counts something else). The phrase does occur in this literature at area 30, and there it counts foldings spread over three target shapes rather than four foldings of one shape. Xu, Horiyama, Shirakawa and Uehara [12] show that nine of the 1080 polyominoes of area 30 folding into $1 \times 1 \times 7$ and $1 \times 3 \times 3$ also fold, along diagonal creases, into a cube of edge $\sqrt{5}$, and that "among these nine polyominoes, one can fold into the third box in two different ways"; their Fig. 5 caption then reads "The unique polygon folds into three boxes of size (a) $1 \times 1 \times 7$, (b) $1 \times 3 \times 3$, and (c)(d) $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$ in four different ways." Mizunashi, Horiyama and Uehara [6] correct the count to

> Among these nine, two polyominoes have four different ways of folding into three different boxes, and seven polyominoes have three (unique) different ways of folding into three different boxes.

The four is $1+1+2$ over three *distinct* shapes, one of which is not an axis-aligned box at all; the largest multiplicity into a single shape there is two. Uehara's book records the same example in the same words – "four different ways to fold three boxes" [11, p. 46], with the doubled shape named in the captions of its Figs. 3.18 and 3.19. Theorem 17 counts foldings of one net into one box, and no source we found states such a count above three below area 28.

6.3. **Four ways: the bound.**

Theorem 19. (i) *Every net of the cube folds into it in exactly one way.*

(ii) *No net of $1 \times 1 \times 2$ folds into it in more than three ways, and three is attained.*

(iii) *Consequently, if $s(B) < 14$ and P folds into B, then* $\mathrm{ways}(B, P) < 4$.

(iv) *Hence the smallest surface area admitting a polyomino that folds into its box in four essentially different ways is* 14 *or* 16.

Proof. (i) and (ii) are exhaustions over the enumerations of Section 5, which are complete by Theorem 10, applicable to every net by Proposition 12, and read one orbit at a time by Proposition 13. For the cube, the enumeration returns 384 tables at its last level and every one has fold count $24 = |\,\mathrm{Rot}\,|$, so ways ≤ 1. For $1 \times 1 \times 2$, it returns 12,124 tables and none accepts more than three of the five orbit representatives, so $|\,\mathrm{Fold}\,| \leq 3 \cdot 8 = 24$ and ways ≤ 3; three is attained by 108 of the tables. (iii) follows because, by Proposition 4, the only boxes of surface area below 14 are the cube and $1 \times 1 \times 2$. (iv) is (iii) together with the area-16 witness of Theorem 17. $\square$

Part (i) is not a finding but a control. It is classical – "each of them is folded in a unique manner" [11, p. 39] – and in any case the cube's rotation group has order 24 and acts simply transitively on its 24 frames, so every cube net folds one way by the two-line argument of Theorem 14(i). Reaching the same conclusion by exhausting 384 developments is a check on the enumeration, and is presented as one. The finding is part (ii), and with it (iii) and (iv).

Corollary 20. *No polyomino of surface area below* 14 *folds into its box in five essentially different ways.*

This is immediate from Theorem 19(iii); we record it because the only quantitative fact known about five-way nets is Tadaki–Amano's area-52 example, which is an upper bound on the smallest five-way area [9].

Remark 21. The one case between the bound and an exact answer is area 14, i.e. the box $1 \times 1 \times 3$. It is a compute question, not a mathematical one: the same enumeration applies verbatim, and an independent implementation of Definition 1 outside the proof assistant reports that the maximum there is again three. Inside the kernel the run needs 238,496 leaves tested against 7 orbit representatives each, which at the measured rate of the $1 \times 1 \times 2$ run projects to roughly ten to twelve CPU-hours; we did not run it. Two routes would close it cheaply: compiling the evaluated code to a shared library, or pruning at a shallow level, since a net with four orbit-distinct root frames has four orbit-distinct partial root frames at every earlier level and level 7 of the area-14 tree holds only about 3,100 cell sets.

6.4. **What is computed.** Since the two exclusion theorems have different characters, we say precisely which claims rest on a finite computation.

Theorem 14(i) rests on five finite checks, none of which mentions a polyomino: $\mathcal{B}(s) = \emptyset$ for $s \in \{7, 8, 9\}$ together with $\mathcal{B}(6) = \{1\times1\times1\}$, and the frame count $|S(1\times1\times1)| = 24$, all three by kernel reduction; and property (G) for the cube together with $|\operatorname{Rot}(1\times1\times1)| = 24$, which are the only two compiled computations in that proof. The rest of it is inequalities. Part (ii) adds one more compiled computation, the fold count of a single 10-cell net.

Theorem 19 rests on exactly two enumeration runs, plus per-box finite checks. The runs are: *the cube* – six levels of $\mathcal{G}$, last level 384 tables, each tested against all 24 frames, all fold counts equal to 24; and $1 \times 1 \times 2$ – ten levels, last level 12,124 tables, each tested against the 5 orbit representatives, with 10,032 tables accepting one orbit, 1,984 accepting two, 108 accepting three and none accepting more. The per-box checks are property (G), the group order $|\operatorname{Rot}(B)|$, the transitivity used in Proposition 12, and the covering condition of Proposition 13; the arithmetic input is that $\mathcal{B}(s) = \emptyset$ for $s \in \{7, 8, 9, 11, 12, 13\}$ and that $\mathcal{B}(6)$, $\mathcal{B}(10)$ are the two singletons named. Nothing about a polyomino enters except through those two runs and the one 16-cell witness; the enumeration layer cost about fourteen CPU-minutes in all. Each witness of Theorems 14(ii) and 17 rests on one fold-count computation over all $4\,s(B)$ frames, plus a canonical-form comparison for non-congruence.

Not claimed as theorems: that the enumeration lists no cell set twice (Section 8), and any statement about area 14 or about uniqueness at area 16 (Remark 31).

7. Negative controls

A decision procedure that always answered "yes" would prove every positive statement here, and a too-permissive definition of folding would prove several. We therefore record what is refuted.

Proposition 22. *The following are theorems: a* 21*-cell polyomino does not fold into* $1 \times 1 \times 5$ *and a* 23*-cell one does not fold into* $1 \times 2 \times 3$ *(both by Proposition 2 alone, with no search); the area-*22 *common unfolding printed in the code repository accompanying* [7]*, with one cell moved – still a* 22*-cell polyomino – does not fold into either box of area* 22*, and likewise one of the area-*106 *nets of* [2] *with one cell moved does not fold into* $1 \times 1 \times 26$*; and there is a* 22*-cell polyomino which folds into* $1 \times 1 \times 5$ *and provably does not fold into* $1 \times 2 \times 3$*, so equal surface area is necessary but not sufficient.*

Proposition 23. *The following are theorems.*

(i) *Of the* 24 *lattice rotations only* 8 *preserve* $1 \times 1 \times 2$ *and only* 4 *preserve* $1 \times 2 \times 3$*, so the filter defining* Rot *is neither everything nor trivial.*
(ii) *"Every net folds in exactly one way" is false; without this the whole layer would be vacuous.*
(iii) *"The number of foldings always equals* $|\operatorname{Rot}|$*" is false:* W_4 *has four times as many.*
(iv) W_4 *does* not *reach five ways.*
(v) *The hypothesis of Theorem 14(i) is not vacuous: the Latin cross is a net of the cube, with* $\mathrm{ways} = 1$.
(vi) *The* 2×5 *rectangle is a* 10*-cell polyomino that provably does not fold into* $1 \times 1 \times 2$ *– a theorem rather than a failed search, by Proposition 3.*

Proposition 24. *The following are theorems: some enumerated development of* $1 \times 1 \times 2$ *accepts exactly three orbits, so the bound of Theorem 19(ii) is tight and "no net of* $1 \times 1 \times 2$ *folds three ways" is false; the enumeration reaches its last level nonempty; no enumerated entry has fold count zero, so the completeness argument is not carried by junk entries; and the witness* W_2 *listed in reverse is a polyomino on which the decision procedure of Proposition 3 returns* false *while the re-rooting procedure of Proposition 9(i) returns true – a concrete listing separating the two, and the reason Lemma 8 is needed.*

8. The definition, checked against published enumerations

Definition 1 is a definition, so whether it is *the* definition can only be settled by agreement with what is already known. We record the agreements here, with the arithmetic (Proposition 28) that gives the open question of [2] its precise shape. None of the numbers below is claimed as new.

Proposition 25. *Exactly* 11 *of the* 35 *free hexominoes fold into the cube, and the other* 24 *provably do not.*

The hexominoes are generated and proved pairwise non-congruent rather than transcribed; the negative half exists only because of Proposition 3.

Proposition 26. *Each of the* 2263 *polyominoes of Uehara's published enumeration* [10] *is a common unfolding of* $1 \times 1 \times 5$ *and* $1 \times 2 \times 3$*, and they are pairwise non-congruent. Moreover* $\mathcal{B}(s)$ *has fewer than two elements for every* $s < 22$*, so* 22 *is the smallest area carrying two boxes.*

The count 2263 is that of [1]; 4526 foldings are verified, with no frames stored – the procedure searches all $4 \cdot 22$ of them per net and box.

Proposition 27. $\mathcal{B}(106) = \{1{\times}1{\times}26,\ 1{\times}2{\times}17,\ 1{\times}5{\times}8\}$*, and each of the* 40 *nets of* [2] *folds into all three; they are pairwise non-congruent. In particular there is a polyomino of area* 106 *folding into* every *box of surface area* 106*.*

That is 120 foldings, each with an explicit frame as certificate; the coordinates are not printed in [2] and were recovered from the data deposit it cites [13].

Proposition 28. *Assume* $\Psi(3) > 58$*, as proved in* [7]*. Then, given the area-*106 *net of Proposition 27, the surface areas at which the smallest three-box common unfolding could still occur are exactly*

$$62,\ 64,\ 70,\ 78,\ 82,\ 88,\ 90,\ 94,\ 96,\ 102,$$

carrying 37 *boxes in all. Furthermore the areas at most* 58 *carrying three boxes are exactly* 46*,* 54 *and* 58*; the first area carrying four boxes is* 70 *and the first carrying five is* 94*.*

This is arithmetic on $s = 2(ab+ac+bc)$, offered as the precise shape of the open question rather than as a result; [2] names the same range in prose.

Proposition 29. *The following are theorems, with the polyominoes read off the figures of* [9]*: a* 28*-cell net folds into* $1 \times 2 \times 4$ *in* 4 *ways; a* 40*-cell net folds into* $1 \times 2 \times 6$ *in* 3 *ways and into* $2 \times 2 \times 4$ *in* 1 *way, hence is a common unfolding of the two boxes of area* 40*; a second* 40*-cell net folds into* $1 \times 2 \times 6$ *in* 4 *ways; a* 52*-cell net folds into* $1 \times 2 \times 8$ *in* 5 *ways; and an* 88*-cell net folds into* $1 \times 4 \times 8$ *in* 4 *ways.*

Five of these are the fold counts printed in the figure captions of [9], including their record; the sixth, one way into $2\times2\times4$, makes precise the same caption's "and also a box of size $2 \times 2 \times 4$". Theirs are counts of crease patterns, computed by the algorithm of [6], which they apply for exactly that purpose ("we apply an algorithm developed in [...] to count the actual number of ways of folding a box"); ours are orbit counts of $\mathrm{Rot}(B)$ on $\mathrm{Fold}(B, P)$. Five agreements between two notions arrived at independently, on nets of four different surface areas, are our evidence that ways is the quantity they are counting; the identification is *not* proved, since the crease-pattern direction needs the three-dimensional realisation we do not formalise, and [9] does not define "different foldings" in the text at all. Finally, [9] publishes no coordinates, so the cell lists were recovered from its figures by grid detection on the electronic version; the transcription is bounded by two checks (cell count equals surface area, fold count equals caption), but the *attribution* of these nets to those figures rests on it. The theorems do not: they say that these polyominoes, whatever their origin, fold in exactly these numbers of ways.

Proposition 30. *Rooted at one fixed frame, the enumeration of Section 5 returns* 384 *tables for the cube – all with fold count* 24 *– and* 12,124 *tables for* $1 \times 1 \times 2$*, split* 10,032/1,984/108 *over fold counts* 8/16/24*. These are the* #SAT *entries of* [7] *for those two boxes.*

What is a theorem here is that no development is missed (Theorem 10). That the level lists contain no cell set twice is *not* proved: the level merge drops adjacent equal keys after a sort, which is exact in practice but would need that comparison proved transitive and total to be a theorem. So "the list has 12,124 entries" is an agreement of numbers between two unrelated pipelines – a SAT formula over cut edges on one side, forced-frame growth on the other – and not a count theorem. Nothing in Theorem 19 uses the count, only that every net occurs.

Remark 31. Outside the formal development we also ran a complete multiplicity census, by two mutually independent programs written from Definition 1, over every net of every box of surface area at most 18 and of $1\times1\times5$ at area 22: 3,353,080 nets in all. Writing the distribution of ways over the nets of a box as a multiset, it comes out as {1:11} for $1\times1\times1$; {1:651, 2:68, 3:4} for $1\times1\times2$; {1:14,931, 2:41, 3:6} for $1\times1\times3$; {1:91,331, 2:2,989, 3:70, 4:1} for $1\times2\times2$; {1:228,343, 2:195, 3:6, 4:3} for $1\times1\times4$; and {1:3,014,241, 2:180, 3:9} for $1\times1\times5$. These figures say that the smallest four-way area is 16 and the net unique there, that exactly three four-way nets exist at area 18, and that no net of area at most 18 folds five ways. *None of this is proved.* It rests on the completeness of a program rather than of a theorem, and is reported here only to say what we expect the answer left open by Theorem 19(iv) to be. The net counts 11/723/14,978/228,547 in these distributions reproduce the simple-net column of [7], and the $1 \times 1 \times 5$ row reproduces the count 3,014,430 published in [2]; the count 94,391 for $1 \times 2 \times 2$ we were unable to locate in the literature. The divisibility of Theorem 6 was asserted for all 3,353,080 nets by a program that knows nothing of its proof, and never failed.

9. Verification

Every theorem, proposition, lemma and definition stated above is formalised and machine-checked in Lean 4 [4] against Mathlib [5], with no `sorry` anywhere; the corollary to Theorem 19 is the immediate weakening of its part (iii). The definitional layer (Definition 1 and Proposition 3), the counting layer (Propositions 5, 13, Theorem 6), the re-rooting lemma 8 with Proposition 9, and the completeness of the enumeration (Theorem 10, Corollary 11) are kernel-pure: they depend only on `propext`, `Classical.choice` and `Quot.sound`. The finite computations – the per-box checks of property (G), the group orders, the frame-transitivity and orbit-covering conditions, the fold counts of the individual witnesses, and the two enumeration runs behind Theorem 19 – are discharged by Lean's compiled evaluator (`native_decide`), each contributing one additional axiom naming that computation. The extremal statement Theorem 19(iv) names ten such axioms: seven per-box facts (two rotation groups, two group orders, two frame-transitivity checks and one orbit-representative check), the two enumeration runs, and the fold count of the 16-cell witness. Statements proved by kernel reduction alone (`decide`) – among them $\mathcal{B}(s) = \emptyset$ for $s \in \{7, 8, 9, 11, 12, 13\}$ – add no axiom. Repository reference to be added at submission.

10. Provenance, and the limits of the search

The claims of priority in Theorems 17 and 19 rest on a literature search, and a search has edges. (Theorem 14 makes none: its conclusion is attributed in Remark 15.) What was swept: a local full-text corpus of 881,145 arXiv papers through late August 2026, under six pattern families (the box-folding vocabulary, the bare and comma-separated forms of every relevant count, the multiplicity and minimality phrases, and the intersection of the 1,246 papers mentioning polyominoes with the 5,487 mentioning a proof assistant); OEIS; the open web; the full text of [9], [7], [2], [6] and the conference version of [12]; and [11], as described below. Within that sweep: the only prior statement of a smallest area for same-box fold multiplicity is [11, p. 39], recorded in Remark 15; no source gives a fold-multiplicity distribution for any box; no source says anything about four foldings of one net into one box, the published "four different ways" at area 30 being a count across three shapes (Remark 18); and [9] – the only paper found that searches for nets with several foldings – works at areas 28, 40, 52 and 88 with a sampling method and makes no minimality claim. Every multiplicity-adjacent sentence in [7] is about non-isomorphic boxes obtained from one net, not about foldings of one net into one box. OEIS contains no sequence for the number of nets of a box.

What was read of Uehara's book. [11] is the one book-length treatment of this subject, and it was reached only through the publisher's preview of the e-book (ISBN 978-981-15-4470-5) on Google Books. Of its Chapter 3, "Common Nets of Boxes" (pp. 25–58), pages 26–29, 38–39 and 53–56 could be displayed and were read in full; pages 25, 30–37, 40–52, 57 and 58 could not be displayed, and the Japanese edition was not consulted. A full-text phrase search over the whole volume was also available, and it reaches pages the preview does not display; roughly thirty phrases were run through it. What that turned up is this. The one passage bearing on Theorem 14 is [11, p. 39], quoted in Remark 15. One further passage concerns several foldings of one shape: at area 30, [11, Figs. 3.18–3.19, p. 48] print nets that fold one of three shapes "in two different ways", a result of [12]. Nothing in the book concerns four foldings of one net into one box: "four ways", "in four" and "different foldings" return nothing at all; "four different" returns [11, p. 46], "four different ways to fold three boxes", which counts foldings across three *distinct* boxes; and "four or more" returns [11, p. 56], "four or more different boxes can be folded", again about distinct boxes. The book contains no count of the nets of a $1 \times 1 \times n$ box and no group-theoretic treatment of "different foldings"; it uses the phrase "essentially different" only of stamp folding. Pages we could not display remain a gap, but one narrowed by the phrase search.

(In passing: [2] reports nets with holes admitting two distinct ways to cut, and writes that their existence "answers the open question posed in Section 3.1.1" of [11]. That book has no Section 3.1.1; what is meant is its Open Problem 3.1.1, on p. 28, which asks "whether or not different boxes can be folded in two or more different cutting ways for given polygons with holes". No polyomino in the present paper has a hole, so that is not the question settled here.)

What was *not* swept, and should be checked by a referee before any statement here is read as a priority claim:

- the pages of [11] listed above as undisplayable were seen only through phrase-search snippets, and the Japanese edition not at all;
- the proceedings of CCCG, WALCOM and JCDCG[3] and the journal *Computational Geometry* were not swept;
- the Japanese-language literature of this subfield was not swept;
- the live arXiv interface was unavailable during the search (it returned HTTP 429 throughout), so coverage of the most recent announcements rests on the local corpus alone; two of six OEIS queries returned HTTP 502, though the four that bear on the counts above completed;
- the data deposits behind [2] may carry per-net folding data not discussed in that paper.

We state these rather than resolve them. A claim of the form "not found" is what a search yields; "new" is what a referee concludes.

References

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[10] Ryuhei Uehara, *Polygons Folding to Plural Incongruent Orthogonal Boxes*, JAIST, electronic data, `https://www.jaist.ac.jp/~uehara/etc/origami/nets/index-e.html`; the 2263 cell lists are the page *All common developments of two boxes of size 1x1x5 and 1x2x3*, `.../nets/all-22.html`. Accessed 30 August 2026.

[11] Ryuhei Uehara, *Introduction to Computational Origami: The World of New Computational Geometry*, Springer Singapore, Singapore, 2020, xviii+220 pp., ISBN 978-981-15-4469-9 (hardcover), 978-981-15-4470-5 (e-book), doi:10.1007/978-981-15-4470-5. Consulted through the publisher's preview only; see §10 for exactly which pages.

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