THE EIGENVECTOR INEQUALITY FOR ALGEBRAIC BOCHNER TENSORS OF EL-HASAN, LI, NIENHAUS, PETERSEN, STANFIELD AND WINK HOLDS WITH THE SHARP CONSTANT 2 IN COMPLEX DIMENSION TWO

KOREA SUPERINTELLIGENCE LABS

Abstract. In arXiv:2608.23778, El-Hasan, Li, Nienhaus, Petersen, Stanfield and Wink prove a Tachibana-type characterisation of complex projective space among Kähler–Einstein manifolds by a partial-positivity condition on the Calabi curvature operator. The constant in their condition comes from an estimate $|SB|^2 + 4|B(S)|^2 \leq c\,|B|^2|S|^2$ for an algebraic Bochner tensor B and an eigenvector S of its Calabi operator; they prove $c \leq 10/4$ and conjecture (their Conjecture 5.2) that $c = 2$ works in every complex dimension. We prove the conjecture in complex dimension two, with the constant 2, and show that the constant is attained. The mechanism is an identity the source does not state: $|SB|^2 = 8|B(S)|^2$ for *every* algebraic Bochner tensor on $\mathbb{C}^2$ and *every* $S \in S^{2,0}$, with no eigenvector hypothesis. It rests on the fact that the Calabi operator carries the five-dimensional space of algebraic Bochner tensors on $\mathbb{C}^2$ onto the trace-free real symmetric 3×3 matrices, which we write down explicitly without square roots; it fails at $n = 3$ (the quadric on $\mathbb{C}^3$ has ratio 4), and, by padding, in every higher dimension. For an eigenvector with eigenvalue τ the left-hand side is therefore exactly $12\tau^2|S|^2$, and the conjecture becomes the eigenvalue bound $3\tau^2 \leq 2\,\mathrm{tr}(C^2)$ for trace-free symmetric 3×3 matrices, which we prove by a Cauchy–Schwarz argument in the Frobenius inner product. Equality holds exactly when the two remaining Calabi eigenvalues coincide, which is the case of the complex quadric $Q^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$; padding that example by zeros shows the constant 2 is attained in complex dimension three as well. Complex dimension $n \geq 3$ remains open; a numerical search over the 27-dimensional space of algebraic Bochner tensors on $\mathbb{C}^3$ found no ratio above 2. Every theorem and proposition below is verified in Lean 4 against *Mathlib*, with no compiled evaluation; the explicit real model and the spectral form of the equality case are stated in prose, and the numerical work is labelled as such.

1. Introduction

Let (M, g) be a Kähler manifold of complex dimension n with Kähler curvature tensor $R_{\alpha\bar\beta\gamma\bar\delta}$ in a unitary frame. The *Calabi curvature operator* is the self-adjoint map $C\colon S^{2,0} \to S^{2,0}$ on symmetric tensors of type $(2,0)$ given by $C(h)_{\bar\beta\bar\delta} = R_{\alpha\bar\beta\gamma\bar\delta}h_{\bar\alpha\bar\gamma}$. It was introduced by Calabi and Vesentini [5] in the deformation theory of compact locally symmetric Kähler manifolds, and on $\mathbb{CP}^n$ with the Fubini–Study metric it is a multiple of the identity. El-Hasan, Li, Nienhaus, Petersen, Stanfield and Wink [1] prove for it a theorem of Tachibana type [8] (their Theorem A): if (M, g) is closed and Kähler–Einstein with $\mathrm{Ric} = \lambda g$, $\lambda \geq 0$, $n \geq 2$, and $C - \frac{\lambda}{n+1}\,\mathrm{id}$ is $\frac{2(n+1)}{5}$-nonnegative, then (M, g) is isometric to $\mathbb{CP}^n$ with the Fubini–Study metric or flat. The paper presents this as the resolution of Problem 2.3 of the AIM problem list *The Bochner technique* [2]. Throughout, statement numbers of [1] are those printed in its compiled version 1.

The constant $\frac{2(n+1)}{5}$ is not expected to be optimal, and the paper isolates exactly where it comes from. Its Theorem B is a Bochner formula whose curvature term, regrouped as in the proof of its Lemma 4.3, is a weighted sum over a unitary eigenbasis $\{S_A\}$ of the Calabi operator of the quantities $|S_AB|^2 + 4|B(S_A)|^2$, where B is the Bochner curvature tensor, $B(S) = C_B(S)$ is the Calabi operator of B applied to S, and SB is the action of $S \in S^{2,0}$ on the $(0,4)$-tensor B (its Definition 2.4). Lemma 4.3 of [1] then reads: put

$$c = \max_{|S|=1} \frac{|SB|^2 + 4|B(S)|^2}{|B|^2}; \tag{1}$$

if $C - \frac{\lambda}{n+1}$ is $\frac{n+1}{c}$-nonnegative then the curvature term is nonnegative. Its proof uses the quotient only on the eigenvectors of C_B (the "highest weight" of the weight principle of [4]), so everything hinges on the best constant c in the purely algebraic inequality

$$|SB|^2 + 4|B(S)|^2 \;\le\; c\,|B|^2|S|^2 \qquad (B \text{ an algebraic Bochner tensor, } S \text{ an eigenvector of } C_B). \tag{2}$$

The source proves

$$|SB|^2 \le 8\,|S|^2|B_{\alpha\bar\beta\gamma\bar\delta}|^2 \;\; \text{(Lemma 4.4)}, \qquad |SB|^2 \le 6\,|S|^2|B_{\alpha\bar\beta\gamma\bar\delta}|^2 \;\; \text{(Lemma 4.5)},$$

shows in Remark 4.6 that 6 is attained on algebraic Kähler curvature tensors, and combines the latter with the operator bound $|B(S)|^2 \le |B_{\alpha\bar\beta\gamma\bar\delta}|^2|S|^2$ and $|B|^2 = 4|B_{\alpha\bar\beta\gamma\bar\delta}|^2$ into $c \le \frac{6+4}{4}$ (in the proof of its Corollary 4.8, whose statement is "Theorem A is true"). It then computes the left side of (2) on the eigenspaces of the Calabi operator of the complex quadrics and of products of projective spaces (its Table 1); the largest ratio there is $\frac{n+2}{n}$, on the line spanned by the real structure q of the quadric Q^n, and the paper states:

> **Conjecture 5.2.** If B is an algebraic Bochner operator, then $|SB|^2 + 4|B(S)|^2 \le 2|B|^2|S|^2$ for any eigenvector S of B, viewed as an operator on $S^{2,0}$.

followed by "No better constant can asymptotically exist based just on algebraic curvature tensor computations, given the examples based on remark 4.6." At $n = 2$ the value $\frac{n+2}{n}$ of Table 1 equals 2, and the text after the conjecture notes that among products of projective spaces the constant is strictly worse "with the exception of $\mathbb{CP}^1 \times \mathbb{CP}^1$, where the constants are equal." A proof of the conjectured value $\frac{n+2}{n}$ for manifolds would, through Lemma 4.3, upgrade Theorem A to the paper's Conjecture 5.3, a sharp characterisation of $\mathbb{CP}^n$ of the kind asked for in Problem 1.9(b) of [2].

Results. We settle the conjecture in complex dimension two, with the conjectured constant, and we show that nothing smaller works. Everything is multilinear algebra on $\mathbb{C}^n$; no manifold appears.

(i) (Theorem 4.1) For every algebraic Bochner tensor B on $\mathbb{C}^2$ and every $S \in S^{2,0}$, with no eigenvector hypothesis,
$$|SB|^2 = 8\,|B(S)|^2.$$
The identity is special to $n = 2$: for the Bochner tensor of the quadric Q^3 and $S = q$ the ratio $|SB|^2/|B(S)|^2$ is 4 (Proposition 4.3).

(ii) (Theorem 5.2) Consequently, if S is an eigenvector with eigenvalue τ, then $|SB|^2 + 4|B(S)|^2 = 12\tau^2|S|^2$ exactly, and equality in (2) with $c = 2$ holds if and only if $3\tau^2 = 2|B_{\alpha\bar\beta\gamma\bar\delta}|^2$.

(iii) (Theorem 5.1) Conjecture 5.2 of [1] holds on $\mathbb{C}^2$: $|SB|^2 + 4|B(S)|^2 \le 2|B|^2|S|^2$ for every algebraic Bochner tensor B on $\mathbb{C}^2$ and every eigenvector S of C_B.

(iv) (Propositions 6.1, 6.2, 6.3) The constant 2 is attained, by the Bochner tensor of $Q^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ with $S = q$, on integer data ($256 + 4 \cdot 32 = 384 = 2 \cdot 96 \cdot 2$); hence no constant below 2 is valid on $\mathbb{C}^2$. Padding this example by zeros attains the constant 2 on $\mathbb{C}^3$ as well ($1536 = 2 \cdot 384 \cdot 2$), so the conjecture, if true, is sharp in complex dimension three and not only asymptotically.

The reason for (i) is structural. The space of algebraic Bochner tensors on $\mathbb{C}^2$ has real dimension $9 - 4 = 5$, and after an explicit change of coordinates on $S^{2,0} \cong \mathbb{C}^3$ the Calabi operator of such a tensor becomes a real symmetric trace-free 3×3 matrix A; this is the model $SU(2) \to SO(3)$, and we write it without square roots (Lemma 3.3). Both $|SB|^2$ and $|B(S)|^2$ are then Hermitian forms in S, quadratic in A, invariant under $SO(3)$, and such forms are spanned by $\mathrm{tr}(A^2)|S|^2$ and $|AS|^2$; the identity says the first coefficient vanishes. The proof, however, is a direct polynomial identity in eleven real variables (Section 4). Given (i), the conjecture at $n = 2$ is the statement $3\tau^2 \le 2\,\mathrm{tr}(A^2)$ for an eigenvalue τ of a trace-free symmetric 3×3 matrix, which we prove with no spectral theorem (Lemma 3.4). Equality forces the two remaining eigenvalues to coincide, and the spectrum $(\tau, -\tau/2, -\tau/2)$ is that of the quadric Q^2.

The identification of the five-dimensional Bochner space on $\mathbb{C}^2$ with the trace-free symmetric 3×3 matrices should be regarded as classical. Its two halves are in Sitaramayya [6]: that the algebraic Kähler curvature tensors on $\mathbb{C}^n$ are exactly the self-adjoint endomorphisms of $S^{2,0}$ is his Theorem 3.6, that the tensors with vanishing Ricci contraction form an orthogonal summand — the "Bochner–Weyl" part, which he identifies with Bochner's tensor — is his Theorem 5.1, and its real dimension $\left(\frac{n(n+1)}{2}\right)^2 - n^2 = \frac{n^2(n-1)(n+3)}{4}$ is his Corollary 5.14, where he notes that the value at $n = 2$ is 5 and that "in general to give some characterisation of the Weyl subspace … seems quite difficult." It is also the algebraic shadow of the fact that on a Kähler surface the Bochner tensor is the anti-self-dual part of the Weyl tensor, for which see Bryant [7, §2.1.3]: "when $n = 2$, so that the underlying manifold has dimension 4 and is canonically oriented, the Bochner tensor turns out to be W^-, the anti-self-dual part of the Weyl curvature." What we claim as new is the identity (i) with its constant, the exact value (ii), and the proof of the constant 2 at $n = 2$; the value 2 itself is printed in Table 1 of [1] as the ratio of an example.

What remains open. Complex dimension $n \geq 3$. Section 8 records a numerical search over the 27-dimensional space of algebraic Bochner tensors on $\mathbb{C}^3$, which found no ratio above 2 and converged to 2 from below, and explains why the two ingredients of the $n = 2$ proof cannot be recombined linearly at $n = 3$. Also open, within the formal development, is the constant 2 on $\mathbb{C}^2$ for arbitrary S rather than eigenvectors; we note in Section 8 what it reduces to. We do not claim the geometric consequence (Conjecture 5.3 of [1] at $n = 2$): its derivation passes through the Bochner formula on a manifold, and no part of that has been checked here.

Verification. Every theorem and proposition below, and Lemmas 3.2 and 3.4, are formally verified in Lean 4 against *Mathlib* (Section 9, Appendix A). Lemma 3.3 (the real model) and Corollary 5.3 (the spectral form of the equality case) are proved in prose only, and say so where they occur; the numerical reproduction of Table 1 and the $n = 3$ search are external computations and are labelled as such.

2. Preliminaries

We follow the conventions of [1, §2], restricted to a Hermitian vector space $\mathbb{C}^n$ with a unitary basis $Z_1, \dots, Z_n$; Greek indices run over $1, \dots, n$, and all sums are written out (no summation convention).

2.1. Algebraic Kähler curvature tensors and Bochner tensors.

Definition 2.1. An *algebraic Kähler curvature tensor* on $\mathbb{C}^n$ is an array $T = (T_{\alpha\bar\beta\gamma\bar\delta}) \in \mathbb{C}^{n^4}$ with

$$T_{\alpha\bar\beta\gamma\bar\delta} = T_{\gamma\bar\beta\alpha\bar\delta} = T_{\alpha\bar\delta\gamma\bar\beta}, \qquad \overline{T_{\alpha\bar\beta\gamma\bar\delta}} = T_{\beta\bar\alpha\delta\bar\gamma} \qquad \text{for all } \alpha, \beta, \gamma, \delta.$$

Its *Ricci contraction* is $T_{\alpha\bar\beta} = \sum_\gamma T_{\alpha\bar\beta\gamma\bar\gamma}$. An *algebraic Bochner tensor* is an algebraic Kähler curvature tensor with vanishing Ricci contraction.

These are the symmetries listed in [1, §2.2] for the Kähler curvature tensor (the fourth one there, $T_{\alpha\bar\beta\gamma\bar\delta} = T_{\gamma\bar\delta\alpha\bar\beta}$, is the composite of the first two), and the vanishing of the Ricci contraction is what characterises the Bochner tensor B in the decomposition $R = \frac{\sigma}{n(n+1)} I + B +$ (traceless Ricci terms) of [1, §2.2], where $I_{\alpha\bar\beta\gamma\bar\delta} = g_{\alpha\bar\beta} g_{\gamma\bar\delta} + g_{\alpha\bar\delta} g_{\gamma\bar\beta}$ and $\sigma = \sum_\alpha R_{\alpha\bar\alpha}$ is half the Riemannian scalar curvature (the source writes S for it; we reserve S for elements of $S^{2,0}$): the scalar and traceless-Ricci parts carry all of the trace. Indexing rows by unordered pairs $\{\alpha, \gamma\}$ and columns by $\{\beta, \delta\}$, an algebraic Kähler curvature tensor is the same thing as a Hermitian $N \times N$ matrix, $N = \binom{n+1}{2}$ ([6, Theorem 3.6]), and the Bochner condition is a Hermitian $n \times n$ matrix of linear conditions, independent because the Ricci contraction is onto the Hermitian $n \times n$ matrices ([6, Theorem 5.12(b)]), so the space of algebraic Bochner tensors on $\mathbb{C}^n$ has real dimension $N^2 - n^2 = \frac{n^2(n-1)(n+3)}{4}$ ([6, Corollary 5.14]): 5 for $n = 2$, 27 for $n = 3$, 84 for $n = 4$.

Following the norm conventions of [1, §2.1] we write

$$|T_{\alpha\bar\beta\gamma\bar\delta}|^2 = \sum_{\alpha,\beta,\gamma,\delta} |T_{\alpha\bar\beta\gamma\bar\delta}|^2, \qquad |T|^2 = 4\, |T_{\alpha\bar\beta\gamma\bar\delta}|^2,$$

the second being the full tensor norm of the real $(0,4)$-tensor.

2.2. **The Calabi operator.** Let $S_n^{2,0}$ denote the symmetric $n\times n$ complex arrays $s=(s_{\alpha\gamma})$, $s_{\alpha\gamma}=s_{\gamma\alpha}$, with $|S|^2=\sum_{\alpha,\gamma}|s_{\alpha\gamma}|^2$ (ordered pairs). For an array T the *Calabi operator* $C_T\colon S_n^{2,0}\to S_n^{2,0}$ is

$$C_T(s)_{\beta\delta}=\sum_{\alpha,\gamma}T_{\alpha\bar\beta\gamma\bar\delta}\,s_{\alpha\gamma},$$

which is the operator $C(h)_{\bar\beta\bar\delta}=T_{\alpha\bar\beta\gamma\bar\delta}h_{\bar\alpha\bar\gamma}$ of [1, §2.3] written on the components of h. We write $B(S)=C_B(S)$ and $|B(S)|^2=\sum_{\beta,\delta}|C_B(s)_{\beta\delta}|^2$. An *eigenvector* of C_B is a nonzero $S\in S_n^{2,0}$ with $C_B(S)=\tau S$ for a real number τ; since C_B is self-adjoint for an algebraic Kähler curvature tensor, the reality of τ is no restriction, but it is a hypothesis in every statement below.

2.3. **The action of S and its norm.** For $S\in S^{2,0}$ and a $(0,4)$-tensor T, [1, Definition 2.4] defines ST by letting S act as a derivation through the "raising of the first index", $ST=-T(S\cdot,\cdot,\cdot,\cdot)-\cdots-T(\cdot,\cdot,\cdot,S\cdot)$. For an algebraic Kähler curvature tensor, Lemma 2.5 there computes $-(ST)(\bar x,\bar y,\bar z,w)=T(S\bar x,\bar y,\bar z,w)-T(S\bar y,\bar x,\bar z,w)$, and Lemma 2.6 gives the norm as four times the sum over $\alpha,\beta,\gamma,\delta$ of the squared modulus of $T(S\bar Z_\alpha,\bar Z_\beta,\bar Z_\gamma,Z_\delta)-T(S\bar Z_\beta,\bar Z_\alpha,\bar Z_\gamma,Z_\delta)$. Since $S\bar Z_\alpha=\sum_\mu s_{\mu\alpha}Z_\mu$, this is the frame-free expression

$$|ST|^2=4\sum_{\alpha,\beta,\gamma,\delta}\Big|\sum_\mu\big(s_{\mu\alpha}T_{\mu\bar\beta\gamma\bar\delta}-s_{\mu\beta}T_{\mu\bar\alpha\gamma\bar\delta}\big)\Big|^2, \tag{3}$$

which we take as the definition of $|ST|^2$. Lemma 2.6 of [1] then specialises to a frame in which S is diagonal with nonnegative entries ρ_α, where it prints (its equation (2.2))

$$|ST|^2=4\sum_{\alpha,\beta,\gamma,\delta}\big|\rho_\alpha T_{\alpha\bar\beta\gamma\bar\delta}-\rho_\beta T_{\beta\bar\alpha\gamma\bar\delta}\big|^2. \tag{4}$$

The first thing checked in the formal development is that (3) and (4) agree, for every n and every real ρ (no sign condition is needed).

Proposition 2.2 (`actNormSq_diagonal`)**.** *Let T be any array on $\mathbb{C}^n$ and $\rho\in\mathbb{R}^n$. For $S=\sum_\alpha\rho_\alpha Z_\alpha\otimes Z_\alpha$, that is $s_{\alpha\gamma}=\rho_\alpha\delta_{\alpha\gamma}$, the right-hand side of* (3) *equals the right-hand side of* (4).

Proof. For diagonal s the inner sum in (3) collapses to the single term $\mu=\alpha$ in the first summand and $\mu=\beta$ in the second. ☐

2.4. **What is known about the constant.** With these definitions, Conjecture 5.2 of [1] is the inequality $|SB|^2+4|B(S)|^2\le 2|B|^2|S|^2$ for every algebraic Bochner tensor B on $\mathbb{C}^n$ and every eigenvector S of C_B, in every $n\ge 2$. The source proves it with $\frac{10}{4}$ in place of 2 (in the proof of its Corollary 4.8, from Lemma 4.5 and the operator bound, both stated in Section 1), and its Remark 4.7 notes that Lemmas 4.4 and 4.5 hold for every algebraic Kähler curvature tensor, the Bochner condition being unused there. Remark 4.6 gives the tensor with $T_{1\bar 21\bar 2}=\pm 2\,T_{1\bar 12\bar 2}\ne 0$ real, all components not forced by the symmetries equal to zero, and $S=Z_1\otimes Z_1+Z_2\otimes Z_2$, for which the constant 6 of Lemma 4.5 is attained; this tensor is Einstein but not Bochner. In the conventions of Section 2 only one of the two signs attains it (Remark 7.2). Table 1 of [1] lists, for the quadrics Q^n, the products $\mathbb{CP}^{d_1}\times\mathbb{CP}^{d_2}$ and $(\mathbb{CP}^1)^k$ with their symmetric metrics scaled to $\lambda=\frac{n+1}{2}$, the Calabi eigenvalues of B, the values of $|SB|^2$ and the ratios $(|SB|^2+4|B(S)|^2)/(|B|^2|S|^2)$ on each eigenspace. Its quadric block reads: on $\langle q\rangle$ the eigenvalue is $-\frac{(n-1)(n+2)}{2n}$, $|SB|^2=2\frac{(n-1)(n+2)^2}{n^2}$ and the ratio is $\frac{n+2}{n}$; on $\langle q\rangle^\perp$ the eigenvalue is $\frac{1}{n}$, $|SB|^2=2(n+1)-\frac{4n+8}{n^2}$ and the ratio is $2\frac{n^2-2}{n(n-1)(n+2)}$; and $|B|^2=\frac{(n+1)(n-1)(n+2)}{n}$. Section 7 reproduces the table from the definitions.

3. Complex dimension two: the normal form and the real model

3.1. **The normal form.** For $b\in\mathbb{R}$ and $p,c\in\mathbb{C}$ let

$$M(b,p,c)=\begin{pmatrix}-b & p & c\\ \bar p & b & -p\\ \bar c & -\bar p & -b\end{pmatrix}$$

and define the array $B(b,p,c)$ on $\mathbb{C}^2$ by $B(b,p,c)_{\alpha\bar\beta\gamma\bar\delta} = M(b,p,c)_{\pi(\alpha,\gamma),\pi(\beta,\delta)}$, where $\pi(1,1)=1$, $\pi(1,2)=\pi(2,1)=2$, $\pi(2,2)=3$ indexes the unordered pairs. Thus $B(b,p,c)_{1\bar11\bar1}=-b$, $B(b,p,c)_{1\bar11\bar2}=p$, $B(b,p,c)_{1\bar21\bar2}=c$, $B(b,p,c)_{1\bar12\bar2}=b$, $B(b,p,c)_{1\bar22\bar2}=-p$, $B(b,p,c)_{2\bar22\bar2}=-b$, and the remaining components are determined by the symmetries.

Proposition 3.1 (`ofParams_isBochner`, `bochner_eq_ofParams`). (a) *For every $(b,p,c)\in\mathbb{R}\times\mathbb{C}\times\mathbb{C}$ the array $B(b,p,c)$ is an algebraic Bochner tensor on $\mathbb{C}^2$.*

(b) *Every algebraic Bochner tensor T on $\mathbb{C}^2$ equals $B\big(\operatorname{Re}T_{1\bar12\bar2},\, T_{1\bar11\bar2},\, T_{1\bar21\bar2}\big)$.*

In particular the algebraic Bochner tensors on $\mathbb{C}^2$ form exactly the family $\{B(b,p,c) : (b,p,c)\in\mathbb{R}\times\mathbb{C}\times\mathbb{C}\}$, of real dimension five, and the hypotheses of the theorems below are not vacuous.

Proof. (a) The two index symmetries hold because π is symmetric, the conjugation symmetry because M is Hermitian, and the Ricci contraction vanishes because $M_{11}+M_{22}=M_{22}+M_{33}=0$ and $M_{12}+M_{23}=0$. (b) A Hermitian 3×3 matrix has nine real parameters; the Ricci conditions $T_{1\bar1\gamma\bar\gamma}=T_{2\bar2\gamma\bar\gamma}=0$ (real) and $T_{1\bar2\gamma\bar\gamma}=0$ (complex) remove four, and solving them gives the pattern of M. In the formal development the sixteen components of T are matched one by one against those of $B(b,p,c)$ from the symmetries and the three Ricci equations. ☐

Lemma 3.2 (`compNormSq_ofParams`). $|B(b,p,c)_{\alpha\bar\beta\gamma\bar\delta}|^2 = 6b^2+8|p|^2+2|c|^2$*, so* $|B(b,p,c)|^2 = 4(6b^2+8|p|^2+2|c|^2)$.

Proof. The entries M_{11},M_{22},M_{33} occur with multiplicities one, four and one among the sixteen components (the pair $\{1,2\}$ has two orderings), $M_{12},M_{21},M_{23},M_{32}$ each twice, and M_{13},M_{31} once: $b^2+4b^2+b^2+8|p|^2+2|c|^2$. ☐

3.2. **The real model.** The space $S^{2,0}_2$ has complex dimension three. For $s\in S^{2,0}_2$ put

$$W(s)=\big(s_{11}-s_{22},\ 2s_{12},\ -i(s_{11}+s_{22})\big)\in\mathbb{C}^3,$$

and for (b,p,c) put

$$A(b,p,c)=\begin{pmatrix}-b-\operatorname{Re}c & 2\operatorname{Re}p & \operatorname{Im}c\\ 2\operatorname{Re}p & 2b & -2\operatorname{Im}p\\ \operatorname{Im}c & -2\operatorname{Im}p & -b+\operatorname{Re}c\end{pmatrix}\in\operatorname{Sym}^2(\mathbb{R}^3).$$

Lemma 3.3. *Let $B=B(b,p,c)$ and $A=A(b,p,c)$.*

(a) *W is a $\mathbb{C}$-linear bijection $S^{2,0}_2\to\mathbb{C}^3$ with $|W(s)|^2=2|S|^2$.*
(b) *$W(C_B(s))=A\,W(s)$ for every $s\in S^{2,0}_2$.*
(c) *$\operatorname{tr}A=0$ and $\operatorname{tr}(A^2)=|B_{\alpha\bar\beta\gamma\bar\delta}|^2=6b^2+8|p|^2+2|c|^2$.*
(d) *$(b,p,c)\mapsto A(b,p,c)$ is a real-linear bijection of $\mathbb{R}\times\mathbb{C}\times\mathbb{C}$ onto the trace-free real symmetric 3×3 matrices.*

Proof. (a) $|s_{11}-s_{22}|^2+|s_{11}+s_{22}|^2=2|s_{11}|^2+2|s_{22}|^2$ and $|2s_{12}|^2=2(|s_{12}|^2+|s_{21}|^2)$. (b) From the definition of C_B and the matrix M,

$$C_B(s)_{11}=-b\,s_{11}+2\bar p\,s_{12}+\bar c\,s_{22},\quad C_B(s)_{12}=p\,s_{11}+2b\,s_{12}-\bar p\,s_{22},\quad C_B(s)_{22}=c\,s_{11}-2p\,s_{12}-b\,s_{22},$$

and substituting into W gives the three rows of $A\,W(s)$; for instance the first component of $W(C_B(s))$ is $-(b+c)s_{11}+(b+\bar c)s_{22}+4\operatorname{Re}p\,s_{12}$, and so is $(-b-\operatorname{Re}c)(s_{11}-s_{22})+2\operatorname{Re}p\cdot 2s_{12}+\operatorname{Im}c\cdot(-i)(s_{11}+s_{22})$. (c) $\operatorname{tr}A=(-b-\operatorname{Re}c)+2b+(-b+\operatorname{Re}c)=0$ and $\operatorname{tr}(A^2)=(b+\operatorname{Re}c)^2+4b^2+(b-\operatorname{Re}c)^2+2(4(\operatorname{Re}p)^2+(\operatorname{Im}c)^2+4(\operatorname{Im}p)^2)$, which is Lemma 3.2. (d) Given a trace-free symmetric A, read off $b=\frac12A_{22}$, $\operatorname{Re}c=\frac12(A_{33}-A_{11})$, $\operatorname{Im}c=A_{13}$, $\operatorname{Re}p=\frac12A_{12}$, $\operatorname{Im}p=-\frac12A_{23}$; the trace condition makes $A_{11}+A_{33}=-2b$ consistent. ☐

So under W the Calabi operator of every algebraic Bochner tensor on $\mathbb{C}^2$ is a real symmetric trace-free 3×3 matrix, every such matrix occurs exactly once, and the Hermitian norm on $S^{2,0}_2$ is half the standard one on $\mathbb{C}^3$. This is the model $SU(2)\to SO(3)$ acting on $\operatorname{Sym}^2(\mathbb{C}^2)\cong\mathbb{C}^3$; the scalar subgroup of $U(2)$ acts on $S^{2,0}_2$ by a phase and trivially on B, and all quantities below are invariant under it. In the formal development the real and imaginary parts of the three identities in (b) are the six linear relations from which Proposition 3.5 is derived; the statement of Lemma 3.3 itself is not a separate formal theorem.

3.3. **The eigenvalue bound.**

Lemma 3.4 (`symm3_eigen_bound`)**.** *Let A be a real symmetric 3×3 matrix with $\operatorname{tr} A = 0$, and let $x \in \mathbb{R}^3$ be nonzero with $Ax = \tau x$, $\tau \in \mathbb{R}$. Then*

$$3\tau^2 \leq 2\operatorname{tr}(A^2) = 2\big(a_{11}^2 + a_{22}^2 + a_{33}^2 + 2a_{12}^2 + 2a_{13}^2 + 2a_{23}^2\big).$$

Proof. Put $r = |x|^2 > 0$, $N = rA - \tau x x^\mathsf{T}$ and $Q = rI - xx^\mathsf{T}$, and write $\langle X, Y\rangle = \operatorname{tr}(X^\mathsf{T} Y)$ for the Frobenius inner product. Using $x^\mathsf{T} A x = \tau r$ and $\operatorname{tr} A = 0$,

$$\langle N, Q\rangle = -\tau r^2, \qquad \langle N, N\rangle = r^2\big(\operatorname{tr}(A^2) - \tau^2\big), \qquad \langle Q, Q\rangle = 2r^2.$$

Cauchy–Schwarz, $\langle N, Q\rangle^2 \leq \langle N, N\rangle\langle Q, Q\rangle$, reads $\tau^2 r^4 \leq 2r^4(\operatorname{tr}(A^2) - \tau^2)$; divide by r^4. No spectral theorem is used: the formal proof is this computation with Cauchy–Schwarz for nine reals against nine reals (*Mathlib*'s `Finset.sum_mul_sq_le_sq_mul_sq`), the three inner products being verified as polynomial identities modulo the eigenvector equations. □

Equality in Lemma 3.4 holds exactly when N and Q are proportional, that is when A acts as a scalar on $x^\perp$: the two remaining eigenvalues coincide. The lemma is the case $N = 3$ of the elementary bound $\tau^2 \leq \frac{N-1}{N}\operatorname{tr}(C^2)$ for a trace-free self-adjoint operator on an N-dimensional space.

Proposition 3.5 (`eigen_bound_ofParams`)**.** *Let $B = B(b, p, c)$, let $s \in S_2^{2,0}$ with $|S|^2 > 0$, and let $\tau \in \mathbb{R}$ with $C_B(s) = \tau s$. Then $3\tau^2 \leq 2|B_{\alpha\bar\beta\gamma\bar\delta}|^2$.*

Proof. Write $W(s) = x + iy$ with $x, y \in \mathbb{R}^3$. By Lemma 3.3(b), $A\,W(s) = \tau W(s)$ with A real, so $Ax = \tau x$ and $Ay = \tau y$; and $|x|^2 + |y|^2 = 2|S|^2 > 0$, so one of x, y is nonzero. Apply Lemma 3.4 to it and use Lemma 3.3(c). □

4. The identity $|SB|^2 = 8|B(S)|^2$

Theorem 4.1 (`actNormSq_eq_eight`)**.** *For every algebraic Bochner tensor B on $\mathbb{C}^2$ and every $S \in S_2^{2,0}$,*

$$|SB|^2 = 8\,|B(S)|^2.$$

No eigenvector hypothesis is needed.

Proof. By Proposition 3.1(b) we may take $B = B(b, p, c)$. Both sides are real polynomials in the eleven real variables $b, \operatorname{Re} p, \operatorname{Im} p, \operatorname{Re} c, \operatorname{Im} c$ and the real and imaginary parts of s_{11}, s_{12}, s_{22}: the left side is (3) with $n = 2$, sixteen squared moduli, each of a combination of four products, the right side is $8\sum_{\beta,\delta}|C_B(s)_{\beta\delta}|^2$ with the three components displayed in the proof of Lemma 3.3. Expanded, the two polynomials coincide. The formal proof does exactly this: it rewrites B in the normal form, unfolds both sides into real and imaginary parts, and closes with the commutative-ring normaliser `ring`; the identity was first found by the same expansion in a computer-algebra system. Independently of the formal proof, the two sides — each quadratic in (b, p, c) and quadratic in s — were evaluated in exact rational arithmetic at the 15×21 pairs of basis points that span the two bi-quadratic forms, and agree there, which again forces the identity. □

Remark 4.2. In the coordinates of Lemma 3.3 the right-hand side is $4|A\,W(s)|^2$. Both $|SB|^2$ and $|B(S)|^2$ are Hermitian forms in $W(s)$ whose coefficients are quadratic in A, and both are invariant under the simultaneous action of $SO(3)$ on A (by conjugation) and on $W(s)$, because $|SB|^2$ and $|B(S)|^2$ are $U(2)$-invariant by construction. Since $\operatorname{Sym}^2(\operatorname{Sym}^2_0 \mathbb{R}^3)$ contains the trivial representation once and the five-dimensional one once, and no three-dimensional one, the $SO(3)$-invariant Hermitian forms of this type are spanned by $\operatorname{tr}(A^2)|W|^2$ and $|AW|^2$. So an identity $|SB|^2 = \alpha\operatorname{tr}(A^2)|W|^2 + \beta|AW|^2$ is forced, and two evaluations (for instance the quadric data of Proposition 6.1, and the same A with $W = (1, 0, 0)$) give $(\alpha, \beta) = (0, 4)$. This explains the identity; the proof above does not depend on it.

The identity is special to complex dimension two. Let B_3 be the array on $\mathbb{C}^3$ with

$$(B_3)_{\alpha\bar\beta\gamma\bar\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta} - 4\,\delta_{\alpha\gamma}\delta_{\beta\delta},$$

and $q_3 = \sum_\alpha Z_\alpha \otimes Z_\alpha$, i.e. $s_{\alpha\gamma} = \delta_{\alpha\gamma}$. In the normalisation $\mathrm{Ric} = n\,g$ of [1, Example 2.2] the quadric Q^3 has $R = gg + gg - q\bar q$ and Bochner tensor $R - \frac{n}{n+1} I$; B_3 is four times that Bochner tensor, scaled to integer components.

Proposition 4.3 (`identity_fails_dim_three`)**.** *B_3 is an algebraic Bochner tensor on $\mathbb{C}^3$, $C_{B_3}(q_3) = -10\,q_3$, and*

$$|q_3 B_3|^2 = 1200 \neq 2400 = 8\,|B_3(q_3)|^2.$$

Proof. The symmetries and the vanishing Ricci contraction are checked from the Kronecker deltas ($\sum_\gamma (B_3)_{\alpha\bar\beta\gamma\bar\gamma} = (3+1-4)\delta_{\alpha\beta} = 0$). $C_{B_3}(q_3)_{\beta\delta} = \sum_\alpha (B_3)_{\alpha\bar\beta\alpha\bar\delta} = (1+1-12)\delta_{\beta\delta}$. Then $|q_3|^2 = 3$, $|B_3(q_3)|^2 = 100\cdot 3 = 300$, and $|q_3B_3|^2$ is the finite sum (3), evaluated to 1200; in the formal development all four numbers are obtained by unfolding the definitions over $\{1,2,3\}^4$ and normalising the resulting integer arithmetic. $\square$

Padding an algebraic Bochner tensor on $\mathbb{C}^n$ by zeros gives an algebraic Bochner tensor on $\mathbb{C}^{n'}$ for $n' > n$ (the new index contributes nothing to the Ricci contraction), and padding S by zeros changes none of $|S|^2$, $|B(S)|^2$, $|SB|^2$; so the counterexample transports to every $n \geq 3$. The failure is not a near miss: over 20 000 random pairs (B, S) — B uniform on the unit sphere of the Bochner space in the norm $|B_{\alpha\bar\beta\gamma\bar\delta}|^2$, S a standard complex Gaussian in an orthonormal basis of $S^{2,0}_n$ — the ratio $|SB|^2/|B(S)|^2$ was observed to range over $[5.49, 132.87]$ at $n = 3$ and over $[6.51, 64.45]$ at $n = 4$, while at $n = 2$ it was 8 to all printed digits in every sample, as Theorem 4.1 requires.

5. The conjecture in complex dimension two

Theorem 5.1 (`conj52_dim_two`)**.** *Let B be an algebraic Bochner tensor on $\mathbb{C}^2$, let $S \in S^{2,0}_2$ be nonzero, and let $\tau \in \mathbb{R}$ with $C_B(S) = \tau S$. Then*

$$|SB|^2 + 4\,|B(S)|^2 \;\leq\; 2\,|B|^2\,|S|^2.$$

That is, Conjecture 5.2 of [1] *holds in complex dimension two, with the conjectured constant.*

Proof. By Theorem 4.1 the left side is $12|B(S)|^2$, and $|B(S)|^2 = |\tau S|^2 = \tau^2|S|^2$. By Proposition 3.1(b) and Proposition 3.5, $3\tau^2 \leq 2|B_{\alpha\bar\beta\gamma\bar\delta}|^2$, so $12\tau^2|S|^2 \leq 8|B_{\alpha\bar\beta\gamma\bar\delta}|^2|S|^2 = 2|B|^2|S|^2$. $\square$

Theorem 5.2 (`exact_value_dim_two`, `equality_iff_dim_two`)**.** *Let B be an algebraic Bochner tensor on $\mathbb{C}^2$, $S \in S^{2,0}_2$ and $\tau \in \mathbb{R}$ with $C_B(S) = \tau S$.*

(a) $|SB|^2 + 4|B(S)|^2 = 12\,\tau^2\,|S|^2$.

(b) *If $S \neq 0$, then $|SB|^2 + 4|B(S)|^2 = 2|B|^2|S|^2$ if and only if $3\tau^2 = 2|B_{\alpha\bar\beta\gamma\bar\delta}|^2$.*

Proof. (a) is the first line of the previous proof. (b) By (a) the equality reads $(12\tau^2 - 8|B_{\alpha\bar\beta\gamma\bar\delta}|^2)|S|^2 = 0$, and $|S|^2 > 0$. $\square$

So at $n = 2$ the quantity that Lemma 4.3 of [1] only bounds is determined by the eigenvalue, and the conjecture is exactly the eigenvalue bound $3\tau^2 \leq 2\,\mathrm{tr}(C_B^2)$ for the trace-free self-adjoint operator C_B on the three-dimensional space $S^{2,0}_2$. In spectral terms:

Corollary 5.3. *Let B be an algebraic Bochner tensor on $\mathbb{C}^2$ with Calabi eigenvalues τ_1, τ_2, τ_3 (real, $\tau_1 + \tau_2 + \tau_3 = 0$, $\tau_1^2 + \tau_2^2 + \tau_3^2 = |B_{\alpha\bar\beta\gamma\bar\delta}|^2$), and let S be an eigenvector with eigenvalue τ_1. Then equality holds in Theorem 5.1 if and only if $\tau_2 = \tau_3$, i.e. if and only if the spectrum of C_B is $(\tau_1, -\tau_1/2, -\tau_1/2)$.*

Proof. By Lemma 3.3 the eigenvalues of C_B are those of the real symmetric matrix A, with $\mathrm{tr}\,A = 0$ and $\mathrm{tr}\,A^2 = |B_{\alpha\bar\beta\gamma\bar\delta}|^2$. Now $3\tau_1^2 = 2(\tau_1^2 + \tau_2^2 + \tau_3^2)$ iff $\tau_1^2 = 2\tau_2^2 + 2\tau_3^2$ iff $(\tau_2 + \tau_3)^2 = 2\tau_2^2 + 2\tau_3^2$ iff $(\tau_2 - \tau_3)^2 = 0$. (This corollary uses the spectral theorem and is proved here in prose only; the formal statement is Theorem 5.2(b).) $\square$

6. The constant 2 is attained

The equality case of Corollary 5.3 is realised by the complex quadric $Q^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. In the normalisation $\mathrm{Ric} = n\,g$ of [1, Example 2.2] its curvature tensor is $R = gg + gg - q\bar{q}$, with Calabi eigenvalues $2-n = 0$ on $\langle q\rangle$ and 2 on $\langle q\rangle^\perp$, and its Bochner tensor is $R - \frac{\sigma}{n(n+1)}I = R - \frac{2}{3}I = B(\frac{1}{3}, 0, -1)$ ($\sigma = 4$ here), whose Calabi spectrum is $(-\frac{4}{3}, \frac{2}{3}, \frac{2}{3})$. We clear denominators: let $B_2 = B(1, 0, -3)$, so that $M = \left(\begin{smallmatrix} -1 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & -1 \end{smallmatrix}\right)$, and let $q = Z_1 \otimes Z_1 + Z_2 \otimes Z_2$, $s_{\alpha\gamma} = \delta_{\alpha\gamma}$.

Proposition 6.1 (`quadric2_equality`). *B_2 is an algebraic Bochner tensor on $\mathbb{C}^2$ and $C_{B_2}(q) = -4q$, with*

$$|B_2|^2 = 96, \qquad |q|^2 = 2, \qquad |B_2(q)|^2 = 32, \qquad |qB_2|^2 = 256,$$

so that $|qB_2|^2 + 4|B_2(q)|^2 = 384 = 2 \cdot 96 \cdot 2 = 2|B_2|^2|q|^2$. Conjecture 5.2 of [1] *is an equality for the quadric Q^2.*

Proof. $A(1, 0, -3) = \mathrm{diag}(2, 2, -4)$ and $W(q) = (0, 0, -2i)$, so $\tau = -4$; $|B_{\alpha\bar\beta\gamma\bar\delta}|^2 = 6 + 18 = 24$ by Lemma 3.2; $|B_2(q)|^2 = 16 \cdot 2$; $|qB_2|^2 = 8 \cdot 32$ by Theorem 4.1; and $3\tau^2 = 48 = 2 \cdot 24$. In the formal development the eigenvector equation and the four norms are evaluated from the definitions on the integer data. □

The value is the one printed in Table 1 of [1] (ratio $\frac{n+2}{n} = 2$ on $\langle q\rangle$ at $n = 2$); what is new is only the exact integer certificate. It turns the upper bound of Theorem 5.1 into a sharp constant:

Proposition 6.2 (`const_two_sharp`). *Let $\kappa \in \mathbb{R}$. If $|SB|^2 + 4|B(S)|^2 \le \kappa |B|^2 |S|^2$ for every algebraic Bochner tensor B on $\mathbb{C}^2$, every nonzero $S \in S^{2,0}_2$ and every $\tau \in \mathbb{R}$ with $C_B(S) = \tau S$, then $\kappa \ge 2$. In particular the inequality with $\kappa = 1.9$ is false, since $384 > 1.9 \cdot 96 \cdot 2 = 364.8$.*

Proof. Instantiate the hypothesis at $(B_2, q, -4)$ and use Proposition 6.1. □

Finally, attainment persists in complex dimension three. Let $\chi_\alpha = 1$ for $\alpha \in \{1, 2\}$ and $\chi_3 = 0$, and define the array $\tilde{B}_3$ on $\mathbb{C}^3$ by

$$(\tilde{B}_3)_{\alpha\bar\beta\gamma\bar\delta} = \chi_\alpha\chi_\beta\chi_\gamma\chi_\delta \Big[\delta_{\alpha\gamma}\delta_{\beta\delta}\big(\delta_{\alpha\beta} + \delta_{\gamma\beta} + \delta_{\alpha\delta} + \delta_{\gamma\delta} - 6\big) + 2(1 - \delta_{\alpha\gamma})(1 - \delta_{\beta\delta})\Big],$$

which is $2B_2$ on indices in $\{1, 2\}$ and zero whenever an index equals 3 (the bracket is the manifestly symmetric form of the matrix $2M$), and let $\tilde{q} = Z_1 \otimes Z_1 + Z_2 \otimes Z_2$ on $\mathbb{C}^3$, $s = \mathrm{diag}(1, 1, 0)$.

Proposition 6.3 (`pad3_equality`). *$\tilde{B}_3$ is an algebraic Bochner tensor on $\mathbb{C}^3$ and $C_{\tilde{B}_3}(\tilde{q}) = -8\tilde{q}$, with*

$$|\tilde{B}_3|^2 = 384, \qquad |\tilde{q}|^2 = 2, \qquad |\tilde{B}_3(\tilde{q})|^2 = 128, \qquad |\tilde{q}\tilde{B}_3|^2 = 1024,$$

so that $|\tilde{q}\tilde{B}_3|^2 + 4|\tilde{B}_3(\tilde{q})|^2 = 1536 = 2 \cdot 384 \cdot 2$: the constant 2 is attained in complex dimension three.

Proof. Doubling multiplies $|B|^2$, $|B(S)|^2$ and $|SB|^2$ by four and the eigenvalue by two, and padding by zeros changes none of them: $384 = 4 \cdot 96$, $128 = 4 \cdot 32$, $1024 = 4 \cdot 256$, $-8 = 2 \cdot (-4)$. The formal development does not use this argument; it evaluates the symmetries, the Ricci contraction, the eigenvector equation and the four norms of $\tilde{B}_3$ directly on $\{1, 2, 3\}^4$. □

The same padding gives, for every $n \ge 2$, an algebraic Bochner tensor on $\mathbb{C}^n$ and an eigenvector with ratio exactly 2; only the case $n = 3$ is formalised. The source records attainment only at $n = 2$ (Table 1 and the remark on $\mathbb{CP}^1 \times \mathbb{CP}^1$) and calls the constant 2 asymptotically sharp; the padding shows that the conjecture, if true, is sharp in every complex dimension.

7. Reproduction of Table 1 of the source

Before anything was claimed, Table 1 of [1] was recomputed from the definitions of Section 2. One cell is certified formally.

Proposition 7.1 (`quadric3_ratio`). *For the quadric tensor B_3 on $\mathbb{C}^3$ and q_3 of Proposition 4.3,*

$$|B_3|^2 = 480, \qquad |q_3|^2 = 3, \qquad |B_3(q_3)|^2 = 300, \qquad |q_3 B_3|^2 = 1200,$$
$$3\big(|q_3B_3|^2 + 4|B_3(q_3)|^2\big) = 7200 = 5\,|B_3|^2|q_3|^2,$$

so the ratio on $\langle q\rangle$ is $\frac{5}{3} = \frac{n+2}{n}$ at $n = 3$, as printed.

Proof. $|(B_3)_{\alpha\bar\beta\gamma\bar\delta}|^2$: the 81 components are -2 on the three positions $\alpha = \beta = \gamma = \delta$, 1 on the $2 \cdot 6$ positions with $\{\alpha = \beta \neq \gamma = \delta\}$ or $\{\alpha = \delta \neq \gamma = \beta\}$, -4 on the six positions with $\alpha = \gamma \neq \beta = \delta$, and 0 elsewhere: $12 + 12 + 96 = 120$, and $|B_3|^2 = 480$. The rest is Proposition 4.3. □

Remark 7.2 (computation). Outside the formal development, every entry of Table 1 of [1] was reproduced from (3) and the definitions of C_B, $|B|^2$, in exact rational arithmetic. The examples are built as in [1, §2.3, §5]: for the quadric, $R = gg + gg - q\bar q$ rescaled from $\mathrm{Ric} = ng$ to $\mathrm{Ric} = \lambda g$ with $\lambda = \frac{n+1}{2}$ as in the table; for $\mathbb{CP}^{d_1} \times \mathbb{CP}^{d_2}$ and $(\mathbb{CP}^1)^k$, the direct sum $\bigoplus_i \frac{\lambda}{d_i+1} I^{(i)}$ of the Fubini–Study tensors of the factors; and $B = R - \frac{\lambda}{n+1} I$ in each case, which we checked to have vanishing Ricci contraction. On each eigenspace listed in the table the quantity $|SB|^2/|S|^2$ is constant (verified on random rational combinations inside the eigenspace), and its value, the eigenvalue, the ratio $(|SB|^2{+}4|B(S)|^2)/(|B|^2|S|^2)$ and $|B|^2$ agree *exactly* with the printed formulas in all three blocks: the quadric for $n = 2, 3, 4, 5$ (so $|B|^2 = \frac{(n+1)(n-1)(n+2)}{n} = 6, \frac{40}{3}, \frac{45}{2}, \frac{168}{5}$ and the ratios $2, \frac{5}{3}, \frac{3}{2}, \frac{7}{5}$ on $\langle q\rangle$ and $\frac{1}{2}, \frac{7}{15}, \frac{7}{18}, \frac{23}{70}$ on $\langle q\rangle^\perp$), the block $\mathbb{CP}^{d_1} \times \mathbb{CP}^{d_2}$ for $(d_1, d_2) \in \{(1,1), (1,2), (2,2), (1,3), (2,3), (3,3)\}$, and the block $(\mathbb{CP}^1)^k$ for $k = 2, \ldots, 5$. For $\mathbb{CP}^n$ itself, $R = I$ gives $C_R = 2\,\mathrm{id}$ and $B = 0$, as in [1, Example 2.1]. Remark 4.6 of [1] was reproduced as well. For the tensor with $T_{1\bar1 2\bar2} = t$ and $T_{1\bar2 1\bar2} = c$ real, everything else not forced by the symmetries zero, and $S = Z_1 \otimes Z_1 + Z_2 \otimes Z_2$, one finds

$$|ST|^2 = 16(c-t)^2, \qquad |S|^2|T_{\alpha\bar\beta\gamma\bar\delta}|^2 = 8t^2 + 4c^2, \qquad \frac{|ST|^2}{|S|^2|T_{\alpha\bar\beta\gamma\bar\delta}|^2} = \frac{4(c-t)^2}{2t^2 + c^2},$$

whose maximum over c is 6, attained exactly at $c = -2t$ (with $t = 1$: $|ST|^2 = 144$, $|S|^2|T_{\alpha\bar\beta\gamma\bar\delta}|^2 = 24$). So the constant of Lemma 4.5 is attained, but — in the conventions of Section 2 — only for the minus sign: $c = +2t$ gives ratio $\frac{2}{3}$, and the source writes $\pm$. We note in passing that the Bochner part of the -2 example is $T - \frac{t}{3}I = B(\frac{2}{3}t, 0, -2t)$, proportional to the Bochner tensor of the quadric Q^2.

8. What remains open

8.1. **Complex dimension three and beyond.** The two ingredients of the proof of Theorem 5.1 are the identity of Theorem 4.1 and the eigenvalue bound; the first fails at $n = 3$ (Proposition 4.3), so the argument does not extend. The following is what is known to us, all of it outside the formal development.

Numerical evidence. For a fixed algebraic Bochner tensor B on $\mathbb{C}^n$ the worst eigenvector inside each eigenspace of C_B can be found exactly, as the top eigenvalue of the positive semidefinite Hermitian form $S \mapsto |SB|^2$ restricted to that eigenspace; the maximum of the ratio $(|SB|^2 + 4|B(S)|^2)/(|B|^2|S|^2)$ over eigenvectors is then a function of B alone, which was maximised by multi-start hill climbing over the Bochner space (real dimension 5 at $n = 2$, 27 at $n = 3$), coordinatised by a basis orthonormal for $|B_{\alpha\bar\beta\gamma\bar\delta}|^2$, so that $|B|^2 = 4$ on the search sphere:

n	dimension	search	best ratio found
2	5	200 random starts, 4000 hill-climbing steps	1.9999999981
3	27	200 random starts, 4000 hill-climbing steps	1.9995231500
3	27	600 s multi-start, seed 11 (453 starts)	1.9995086884
3	27	600 s multi-start, seed 12 (437 starts)	1.9995431951

The evaluation of the ratio was checked against the two values known in closed form: it returns $\frac{5}{3}$ on the quadric tensor B_3 of Proposition 4.3, the cell of Table 1 certified in Proposition 7.1, and exactly 2 on the padded extremal $\tilde B_3$ of Proposition 6.3. Both dimensions converge to 2 from below and no start exceeded it, while 2 is attained at $n = 3$ by Proposition 6.3. So the conjecture looks true at $n = 3$, and sharp there. A search is of course not a proof; and the normalisation matters, since coordinates on the Bochner space that are not orthonormal for $|B_{\alpha\bar\beta\gamma\bar\delta}|^2$ silently misreport $|B|^2$ and with it the ratio.

Why the obvious repair fails. At $n = 3$ the eigenvalue bound is $\tau^2 \le \frac{5}{6}\operatorname{tr}(C_B^2)$ ($N = \dim S_3^{2,0} = 6$), where $\operatorname{tr}(C_B^2) = |B_{\alpha\bar\beta\gamma\bar\delta}|^2$ (the source notes $\sum_A \tau_A^2 = |B_{\alpha\bar\beta\gamma\bar\delta}|^2$ for the Calabi eigenvalues of a Bochner tensor), and Lemma 4.5 of [1] gives $|SB|^2 \le 6\operatorname{tr}(C_B^2)|S|^2$; added, they give only the source's $\frac{10}{4}$. One might hope for a joint bound $|SB|^2 \le \alpha\operatorname{tr}(C_B^2)|S|^2 + \beta\tau^2|S|^2$ on eigenvectors, which together with the eigenvalue bound gives the conjecture as soon as $\alpha + \frac{5}{6}(\beta + 4) \le 8$. But the padded extremal of Proposition 6.3 has $\operatorname{tr}(C^2) = |B_{\alpha\bar\beta\gamma\bar\delta}|^2 = 96$, $\tau^2 = 64$ and $|SB|^2/|S|^2 = 512$, so any valid pair satisfies $96\alpha + 64\beta \ge 512$, i.e. $3\alpha + 2\beta \ge 16$. Together with $\alpha + \frac{5}{6}(\beta + 4) \le 8$ this forces $\alpha \ge 8$ and $\beta \le -4$, and the bound $|SB|^2 \le 8\operatorname{tr}(C_B^2)|S|^2 - 4\tau^2|S|^2$ is the conjecture itself. No linear combination of a bound on $|SB|^2$ alone and the eigenvalue bound can therefore prove the conjecture at $n = 3$; a genuinely joint certificate is needed.

Cost of the two certificates that would work. A flat Positivstellensatz certificate in the $27+12+1 = 40$ real variables (Bochner tensor, $S \in \mathbb{C}^6$, τ), of degree four, with the twelve real bilinear eigenvector equations as constraints, has a positive semidefinite block indexed by the $\binom{42}{2} = 861$ monomials of degree at most two and $\binom{44}{4} = 135\,751$ coefficient-matching constraints; the Schur complement of an interior-point solve is then of order $135\,751^2 \times 8$ bytes ≈ 147 GB. A stratified route — reduce S to $\operatorname{diag}(\rho_1, \rho_2, \rho_3)$, $\rho \ge 0$, by Takagi's factorisation, for fixed ρ check a quadratic form on the linear subspace of $\mathbb{R}^{28}$ of pairs (B, τ) with $C_B S = \tau S$, and certify it in ρ by a sum-of-squares matrix on each of the six coincidence strata of ρ — has small semidefinite programmes but requires the strata, their degeneracies and the unitary reduction to be organised and formalised, which we estimate as several days of work. Neither was attempted; the three counts 861, 135 751 and 147 GB are arithmetic consequences of the two binomial coefficients and of the dense Schur complement of an interior-point step, not measurements of a run.

8.2. **Arbitrary S in complex dimension two.** Theorem 4.1 holds for every S, so for arbitrary nonzero $S \in S_2^{2,0}$ the inequality $|SB|^2 + 4|B(S)|^2 \le 2|B|^2|S|^2$ is, in the coordinates of Lemma 3.3, $3|AW|^2 \le 2\operatorname{tr}(A^2)|W|^2$, i.e. the positive semidefiniteness of $2\operatorname{tr}(A^2)I - 3A^2$ for trace-free real symmetric 3×3 A. With the spectral theorem this is Lemma 3.4 applied to each eigenvalue of A, so the constant $c = 2$ of (1) at $n = 2$ holds over all of $S_2^{2,0}$; a proof inside the formal development, which avoids the spectral theorem, would need a sum-of-squares certificate for the quartic $2\operatorname{tr}(A^2)|W|^2 - 3|AW|^2$ in the five entries of A and the six real components of W. We have not constructed one, and we therefore state the general-S constant only as a remark.

8.3. **The geometric statement.** Lemma 4.3 of [1] converts a bound on c into a partial-positivity hypothesis on $C - \frac{\lambda}{n+1}$, and Corollary 4.8 there converts the latter into Theorem A by the maximum principle and the tables of [5]. With $c = 2 = \frac{n+2}{n}$ at $n = 2$ this would be the case $n = 2$ of Conjecture 5.3 of [1]. We do not claim it: our results are about eigenvectors of algebraic Bochner tensors only, and nothing in the Bochner-formula argument has been checked here.

9. Verification

Propositions 2.2, 3.1, 4.3, 6.1, 6.2, 6.3 and 7.1, Lemmas 3.2 and 3.4, Proposition 3.5 and Theorems 4.1, 5.1 and 5.2 have been formally verified in Lean 4 against *Mathlib* [9, 10]; their statements are reproduced verbatim in Appendix A. The development is four modules of 729 lines in all, in import order `Linear` (96 lines; Cauchy–Schwarz for nine reals and Lemma 3.4), `Defs` (179 lines; the twelve definitions of Section 2, the three of the normal form, and Proposition 3.1), `Sharp2` (209 lines; eight theorems: Theorem 4.1, Lemma 3.2, $|B(S)|^2 = \tau^2|S|^2$, Proposition 3.5, positivity of $|S|^2$ for $S \ne 0$, Theorem 5.1 and the two parts of Theorem 5.2) and `Examples` (245 lines; eight definitions and thirty-three theorems: Proposition 2.2, the quadric data at $n = 2$ and $n = 3$, the padded tensor, and Propositions 4.3, 6.1, 6.2, 6.3, 7.1). Twelve declarations carry the statements of this paper; a print of the axiom set of each of them, and of the ten auxiliary declarations they rest on (among them `symm3_eigen_bound`, `cs9`, `quadric3_isBochner` and `pad3_isBochner`), returns `propext`, `Classical.choice` and `Quot.sound` and nothing else: there is no `sorry`, no axiom is declared by hand, and no compiled evaluation (`native_decide`) is used anywhere. The identity of Theorem 4.1 is closed by `ring` after unfolding into real and imaginary parts; the statements about the integer witnesses are discharged by unfolding the finite sums and normalising the arithmetic (`simp`, `norm_num`); Lemma 3.4 uses *Mathlib*'s Cauchy–Schwarz inequality for finite sums and the linear-arithmetic and `linear_combination` tactics. Checking the four modules one at a time with `lake env lean`, in import order, on a twenty-core Linux machine carrying other work, took 74, 13, 14 and 9 seconds of wall time (111 s in all) and 39, 13, 19 and 21 seconds of processor time; the first module pays the cold read of the *Mathlib* object files. Peak resident memory was 7.2 GB for a single *Mathlib*-loaded process. The negative controls are those stated above: the too-small constant $\kappa = 1.9$ refuted (Proposition 6.2), the identity refuted at $n = 3$ (Proposition 4.3), the non-vacuity of the hypotheses (Proposition 3.1(a)), and the agreement of the definition of $|ST|^2$ with the source's printed formula (Proposition 2.2) and with a printed cell of its Table 1 (Proposition 7.1).

Outside the formal development, and labelled as such where they occur, are the reproduction of Table 1 and of Remark 4.6 (Remark 7.2; exact rational arithmetic in Python, under a processor-minute), the symbolic discovery of the identity (a computer-algebra expansion in eleven real variables, then re-proved in Lean, and the exact spanning-set check in the proof of Theorem 4.1), and the $n = 3$ search of Section 8 (four runs, about 27 processor-minutes in all, in double precision). The source of the development and of the scripts is currently held in a private repository: repository reference to be added at submission.

On the reference list. All statement numbers attributed to [1] — Examples 2.1–2.3, Definition 2.4, Lemmas 2.5 and 2.6 with its equation (2.2), Lemma 4.3, Lemmas 4.4 and 4.5, Remarks 4.6 and 4.7, Corollary 4.8, Conjectures 5.2 and 5.3 and Table 1 — are those printed in the compiled e-print of version 1: the authors' LaTeX source was compiled with a current distribution, its auxiliary file read for the labelled statements, and the unlabelled ones (Examples 2.1–2.3, Remark 4.7, Corollary 4.8, Conjecture 5.3) read in the resulting PDF. On 7 September 2026 the abstract page listed one version only, submitted 24 August 2026. The bibliographic data of [5], [6], [8], [4] and [3] were checked against Crossref and zbMATH rather than copied from the reference list of [1]; [6] was read in the original.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted), grouped by module. Indices are the elements of `Fin n` $= \{0, \ldots, n-1\}$, so the 1-based components $T_{1\bar{1}2\bar{2}}$, $T_{1\bar{1}1\bar{2}}$, $T_{1\bar{2}1\bar{2}}$ of Proposition 3.1 are `T 0 0 1 1`, `T 0 0 0 1`, `T 0 1 0 1`. `Tens n` is the component array $T_{\alpha\bar{\beta}\gamma\bar{\delta}}$; `IsAlgKahler`, `ricci`, `IsBochner` are Definition 2.1; `compNormSq` and `fullNormSq` are $|T_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2$ and $|T|^2$; `IsSym` and `sNormSq` are membership in $S^{2,0}_n$ and $|S|^2$; `calabi`, `calabiNormSq`, `actNormSq` are $C_T(s)$, $|B(S)|^2$ and $|ST|^2$ as in (3); `IsEigen T s τ` is $C_T(s) = \tau s$; `ofParams b p c` is $B(b, p, c)$; `quadric2`, `qvec` are B_2, q; `quadric3`, `qvec3` are B_3, q_3; `pad3`, `pvec3` are $\tilde{B}_3$, $\tilde{q}$; `Complex.normSq` is $|\cdot|^2$ and `starRingEnd ℂ` is complex conjugation.

Linear.lean

```
theorem symm3_eigen_bound
    (a11 a12 a13 a22 a23 a33 x1 x2 x3 τ : ℝ)
    (htr : a11 + a22 + a33 = 0)
    (h1 : a11 * x1 + a12 * x2 + a13 * x3 = τ * x1)
    (h2 : a12 * x1 + a22 * x2 + a23 * x3 = τ * x2)
    (h3 : a13 * x1 + a23 * x2 + a33 * x3 = τ * x3)
    (hx : 0 < x1 ^ 2 + x2 ^ 2 + x3 ^ 2) :
    3 * τ ^ 2 ≤ 2 * (a11 ^ 2 + a22 ^ 2 + a33 ^ 2 + 2 * a12 ^ 2 + 2 * a13 ^ 2 + 2 * a23 ^ 2)
```

Defs.lean

```
abbrev Tens (n : ℕ) := Fin n → Fin n → Fin n → Fin n → ℂ
structure IsAlgKahler {n : ℕ} (T : Tens n) : Prop where
  swap13 : ∀ a b c d, T a b c d = T c b a d
  swap24 : ∀ a b c d, T a b c d = T a d c b
  herm : ∀ a b c d, (starRingEnd ℂ) (T a b c d) = T b a d c
def ricci {n : ℕ} (T : Tens n) (a b : Fin n) : ℂ := ∑ c, T a b c c
structure IsBochner {n : ℕ} (T : Tens n) : Prop where
  toIsAlgKahler : IsAlgKahler T
```

```
  ricci_zero : ∀ a b, ricci T a b = 0
def compNormSq {n : ℕ} (T : Tens n) : ℝ := ∑ a, ∑ b, ∑ c, ∑ d, Complex.normSq (T a b c d)
def fullNormSq {n : ℕ} (T : Tens n) : ℝ := 4 * compNormSq T
def IsSym {n : ℕ} (s : Fin n → Fin n → ℂ) : Prop := ∀ a c, s a c = s c a
def sNormSq {n : ℕ} (s : Fin n → Fin n → ℂ) : ℝ := ∑ a, ∑ c, Complex.normSq (s a c)
def calabi {n : ℕ} (T : Tens n) (s : Fin n → Fin n → ℂ) (b d : Fin n) : ℂ :=
  ∑ a, ∑ c, T a b c d * s a c
def calabiNormSq {n : ℕ} (T : Tens n) (s : Fin n → Fin n → ℂ) : ℝ :=
  ∑ b, ∑ d, Complex.normSq (calabi T s b d)
def actNormSq {n : ℕ} (T : Tens n) (s : Fin n → Fin n → ℂ) : ℝ :=
  4 * ∑ a, ∑ b, ∑ c, ∑ d,
    Complex.normSq (∑ m, (s m a * T m b c d - s m b * T m a c d))
def IsEigen {n : ℕ} (T : Tens n) (s : Fin n → Fin n → ℂ) (τ : ℝ) : Prop :=
  ∀ b d, calabi T s b d = (τ : ℂ) * s b d
def pr : Fin 2 → Fin 2 → Fin 3
  | 0, 0 => 0
  | 1, 1 => 2
  | _, _ => 1
def mtx (b : ℝ) (p c : ℂ) : Fin 3 → Fin 3 → ℂ :=
  ![![-(b : ℂ), p, c],
    ![(starRingEnd ℂ) p, (b : ℂ), -p],
    ![(starRingEnd ℂ) c, -((starRingEnd ℂ) p), -(b : ℂ)]]
def ofParams (b : ℝ) (p c : ℂ) : Tens 2 := fun i j k l => mtx b p c (pr i k) (pr j l)
theorem ofParams_isBochner (b : ℝ) (p c : ℂ) : IsBochner (ofParams b p c)
theorem bochner_eq_ofParams {T : Tens 2} (hT : IsBochner T) :
    T = ofParams (T 0 0 1 1).re (T 0 0 0 1) (T 0 1 0 1)
```

Sharp2.lean

```
theorem actNormSq_eq_eight {T : Tens 2} (hT : IsBochner T) {s : Fin 2 → Fin 2 → ℂ}
    (hs : IsSym s) : actNormSq T s = 8 * calabiNormSq T s
theorem compNormSq_ofParams (b : ℝ) (p c : ℂ) :
    compNormSq (ofParams b p c) = 6 * b ^ 2 + 8 * Complex.normSq p + 2 * Complex.normSq c
theorem calabiNormSq_of_eigen {n : ℕ} {T : Tens n} {s : Fin n → Fin n → ℂ} {τ : ℝ}
    (h : IsEigen T s τ) : calabiNormSq T s = τ ^ 2 * sNormSq s
theorem eigen_bound_ofParams (b : ℝ) (p c : ℂ) {s : Fin 2 → Fin 2 → ℂ} (hs : IsSym s)
    (hpos : 0 < sNormSq s) {τ : ℝ} (heig : IsEigen (ofParams b p c) s τ) :
    3 * τ ^ 2 ≤ 2 * compNormSq (ofParams b p c)
theorem conj52_dim_two {T : Tens 2} (hT : IsBochner T) {s : Fin 2 → Fin 2 → ℂ}
    (hs : IsSym s) (hs0 : s ≠ 0) {τ : ℝ} (heig : IsEigen T s τ) :
    actNormSq T s + 4 * calabiNormSq T s ≤ 2 * fullNormSq T * sNormSq s
theorem exact_value_dim_two {T : Tens 2} (hT : IsBochner T) {s : Fin 2 → Fin 2 → ℂ}
    (hs : IsSym s) {τ : ℝ} (heig : IsEigen T s τ) :
    actNormSq T s + 4 * calabiNormSq T s = 12 * τ ^ 2 * sNormSq s
theorem equality_iff_dim_two {T : Tens 2} (hT : IsBochner T) {s : Fin 2 → Fin 2 → ℂ}
    (hs : IsSym s) (hs0 : s ≠ 0) {τ : ℝ} (heig : IsEigen T s τ) :
    (actNormSq T s + 4 * calabiNormSq T s = 2 * fullNormSq T * sNormSq s)
      ↔ 3 * τ ^ 2 = 2 * compNormSq T
```

Examples.lean

```
def dl {n : ℕ} (i j : Fin n) : ℂ := if i = j then 1 else 0
theorem actNormSq_diagonal {n : ℕ} (T : Tens n) (ρ : Fin n → ℝ) :
    actNormSq T (fun a c => if a = c then (ρ a : ℂ) else 0)
      = 4 * ∑ a, ∑ b, ∑ c, ∑ d,
          Complex.normSq ((ρ a : ℂ) * T a b c d - (ρ b : ℂ) * T b a c d)
def quadric2 : Tens 2 := ofParams 1 0 (-3)
def qvec : Fin 2 → Fin 2 → ℂ := fun a c => dl a c
theorem quadric2_isEigen : IsEigen quadric2 qvec (-4)
theorem quadric2_equality :
    actNormSq quadric2 qvec + 4 * calabiNormSq quadric2 qvec
      = 2 * fullNormSq quadric2 * sNormSq qvec
theorem const_two_sharp (κ : ℝ)
    (h : ∀ T : Tens 2, IsBochner T → ∀ s : Fin 2 → Fin 2 → ℂ, IsSym s → s ≠ 0 → ∀ τ : ℝ,
      IsEigen T s τ → actNormSq T s + 4 * calabiNormSq T s ≤ κ * fullNormSq T * sNormSq s) :
    2 ≤ κ
def quadric3 : Tens 3 := fun a b c d => dl a b * dl c d + dl a d * dl c b - 4 * (dl a c * dl b d)
def qvec3 : Fin 3 → Fin 3 → ℂ := fun a c => dl a c
theorem quadric3_isEigen : IsEigen quadric3 qvec3 (-10)
theorem quadric3_ratio :
    3 * (actNormSq quadric3 qvec3 + 4 * calabiNormSq quadric3 qvec3)
      = 5 * (fullNormSq quadric3 * sNormSq qvec3)
theorem identity_fails_dim_three :
    actNormSq quadric3 qvec3 ≠ 8 * calabiNormSq quadric3 qvec3
def chi : Fin 3 → ℂ := fun i => if i = 2 then 0 else 1
def pad3 : Tens 3 := fun a b c d =>
  chi a * chi b * chi c * chi d *
    (dl a c * dl b d * (dl a b + dl c b + dl a d + dl c d - 6)
```

```
      + 2 * ((1 - dl a c) * (1 - dl b d)))
def pvec3 : Fin 3 → Fin 3 → ℂ := fun a c => chi a * chi c * dl a c
theorem pad3_isEigen : IsEigen pad3 pvec3 (-8)
theorem pad3_equality :
    actNormSq pad3 pvec3 + 4 * calabiNormSq pad3 pvec3
      = 2 * fullNormSq pad3 * sNormSq pvec3
```

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[2] American Institute of Mathematics, *The Bochner technique*, AIM Problem Lists, workshop of May 2026, `http://aimpl.org/bochner`; read on 7 September 2026. Problem 1.9 (page `http://aimpl.org/bochner/1/`) asks: "Can we characterize Fubini-Study metric $\mathbb{CP}^{n/2}$ among Kähler Einstein manifolds under conditions involving curvature operator of the second kind or Calabi curvature operator? (a) Can we find a condition that makes $\mathbb{CP}^{n/2}$ the only example? (b) Can we find a sharp condition?" (there n is the real dimension). Problem 2.3 (page `http://aimpl.org/bochner/2/`, the page [1] cites) asks: "Is there a Tachibana Theorem that uses Calabi curvature operator?"

[3] K. Broder, J. Nienhaus, P. Petersen, J. Stanfield and M. Wink, *Vanishing theorems for Hodge numbers and the Calabi curvature operator*, Adv. Math. **500** (2026), Paper No. 111077, doi:10.1016/j.aim.2026.111077; arXiv:2503.06870. (The direct predecessor of [1]; its action estimate concerns (p,q)-forms and it contains no estimate of $|SB|^2+4|B(S)|^2$.)

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