POWER SUMS OF THE ZEROS OF THE BLZ POLYNOMIALS: A PROOF OF THE SECOND-POWER-SUM CONJECTURE OF MASOERO AND RUZZA, A CLOSED FORMULA FOR THE THIRD POWER SUM, AND THE MEMBERSHIP HALF OF THEIR QUESTION 3.7

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Abstract. For a partition λ and a parameter β, Masoero and Ruzza (arXiv:2605.24563v2) attach to the Laguerre Wronskian $\Phi^{(\beta)}_{\lambda,\lambda'}$ a monic polynomial $\mathcal{P}^{(\beta)}_{\lambda}$ of degree $|\lambda|$ — the BLZ polynomial whose zeros $z^{(\beta)}_i(\lambda)$ are the poles of the monster potentials of Bazhanov–Lukyanov–Zamolodchikov at the free-fermion point — and prove that the sum of the zeros is the shifted symmetric function $2\mathsf{p}_4-2\mathsf{p}_2^2+3\beta\mathsf{p}_3+\beta^2\mathsf{p}_2+\frac{1}{2}\mathsf{p}_2$ of λ (their Proposition 3.6). For the sum of the squares they conjecture a degree-six shifted symmetric expression (their Eq. (3.8)), "verified for all $\lambda\in\mathbb{P}$ with $|\lambda|\le 15$", and they ask (Question 3.7) whether the k-th power sum lies in $\Lambda^*[\beta]_{\le 2k+2}$ for every k. We prove the conjecture for every partition and every β. In the reversed Wronskian matrix the coefficient polynomial $G_p(X)=(-1)^p\,(X/2)^{\underline{p}}\,(X/2-\beta)^{\underline{p}}/p!$ attached to a column is literally the same for the two kinds of Laguerre column, so the determinant is alternating in the nodes, the Vandermonde divides it, and the k-th subleading coefficient is a symmetric polynomial in the nodes of total degree at most $2k$. This places the difference of the two sides of Eq. (3.8) in $\Lambda^*[\beta]_{\le 16}$, a space pinned down by the partitions of size at most 16: a rank-231 certificate and the exact verification of Eq. (3.8), as an identity of polynomials in β, for every partition of size at most 16 finish the proof. The proof itself is on paper; what is machine-checked is exactly its two finite ingredients — the rank certificate and Eq. (3.8) for $|\lambda|\le 16$ — together with Eq. (3.8) for $|\lambda|\le 17$ and, at five values of β that decide it, for $|\lambda|\le 21$. The same lemma settles, again on paper, the membership half of Question 3.7, $\sum_i (z^{(\beta)}_i)^k\in\Lambda^*[\beta]_{\le 8k}$ for every k; only the sharp bound $2k+2$ remains open. We also give a closed formula for the third power sum, an element of $\Lambda^*[\beta]_{\le 8}$ that the source does not have, found by exact fitting on $|\lambda|\le 10$, verified on all 369 partitions of size 11 to 14 before being written down, and machine-checked for $|\lambda|\le 14$; and we record that the top-degree layer of the k-th power sum is row $2k$ of OEIS A088617 reversed, with the Catalan number C_{2k} in front of p_{2k+2}, for $k=0,1,2,3$.

1. Introduction

1.1. **The objects and the source.** Let $\widehat{L}^{(\beta)}_n(y)$ be the monic generalized Laguerre polynomial and

$$\psi^{(\beta)}_{\pm}(n,x)=e^{-x^2/2}\,x^{1/2\pm\beta}\,\widehat{L}^{(\pm\beta)}_n(x^2). \tag{1}$$

In [1], Masoero and Ruzza study the Wronskians

$$\Psi^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(x)=\mathrm{Wr}_x\big(\psi^{(\beta)}_+(m_1,x),\dots,\psi^{(\beta)}_+(m_r,x),\psi^{(\beta)}_-(n_1,x),\dots,\psi^{(\beta)}_-(n_s,x)\big) \tag{2}$$

of these functions (their Eq. (2.7)), prove that each is an exponential times a power of x times a monic polynomial $\widetilde{\Phi}^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(x^2)$ (their Theorem 2.2), show that the polynomial depends on the strictly decreasing tuples $\boldsymbol{m},\boldsymbol{n}$ only through a pair of partitions (λ,μ) once the parameter is shifted (their Theorem 2.5 and Definition 2.6), and prove that in the symmetric case $\mu=\lambda'$ the polynomial $\Phi^{(\beta)}_{\lambda,\lambda'}(y)$ is even (their Theorem 3.1). This lets them define (their Definition 3.4)

$$\Phi^{(\beta)}_{\lambda,\lambda'}(y)=\mathcal{P}^{(\beta)}_{\lambda}(y^2)=\prod_{i=1}^{|\lambda|}\big(y^2-z^{(\beta)}_i(\lambda)\big). \tag{3}$$

Their Theorem 5.12(VII) identifies $\mathcal{P}^{(\beta)}_{\lambda}(x^4)$ with a BLZ polynomial of level $|\lambda|$ and momentum β whenever β is not a root of $\phi^{[0]}_{\lambda,\lambda'}$: the zeros $z^{(\beta)}_i(\lambda)$ are the poles of the monster potentials of Bazhanov,

Lukyanov and Zamolodchikov [2] at the free-fermion point, and they satisfy the BLZ system, Eq. (5.4) of [1]. (All theorem, equation and page numbers of [1] quoted in this paper were read off the compiled version-2 e-print and its PDF, never counted from the raw source.)

The source then turns to the power sums of the zeros. With the shifted symmetric functions of Kerov and Olshanski [6] in the normalisation of [1, Definition 3.5],

$$\mathsf{p}_k(\lambda) = \sum_{i\geq 1}\Big(\big(\lambda_i - i + \tfrac{1}{2}\big)^{k-1} - \big(-i+\tfrac{1}{2}\big)^{k-1}\Big) \quad (k\geq 1), \qquad \mathsf{p}_0 = 1, \qquad \Lambda^* = \mathbb{Q}[\mathsf{p}_2, \mathsf{p}_3, \dots], \tag{4}$$

they prove (their Proposition 3.6, Eq. (3.5))

$$\sum_{i=1}^{|\lambda|} z_i^{(\beta)}(\lambda) = 2\mathsf{p}_4 - 2\mathsf{p}_2^2 + 3\beta\mathsf{p}_3 + \beta^2\mathsf{p}_2 + \tfrac{1}{2}\mathsf{p}_2, \tag{5}$$

by specialising the closed formula for the second subleading coefficient of $\widetilde{\Phi}$ that their Theorem 2.2 provides. Immediately afterwards (p. 8) they write, verbatim:

> We conjecture that (omitting the dependence on λ)
>
> $$\sum_{i=1}^{|\lambda|}\big(z_i^{(\beta)}\big)^2 = \Big(14\mathsf{p}_6 - 40\mathsf{p}_4\mathsf{p}_2 + \tfrac{64}{3}\mathsf{p}_2^3 + \tfrac{65}{3}\mathsf{p}_4 - 20\mathsf{p}_2^2 + \tfrac{27}{8}\mathsf{p}_2\Big) + \beta\Big(35\mathsf{p}_5 - 60\mathsf{p}_3\mathsf{p}_2 + \tfrac{65}{2}\mathsf{p}_3\Big) + \beta^2\Big(30\mathsf{p}_4 - 22\mathsf{p}_2^2 + \tfrac{25}{2}\mathsf{p}_2\Big) + 10\beta^3\mathsf{p}_3 + \beta^4\mathsf{p}_2\,. \tag{6}$$
>
> This conjecture is based on numerical experiments and is verified for all $\lambda \in \mathbb{P}$ with $|\lambda| \leq 15$. Although a proof using the methods used in Proposition 3.6 seems, in principle, possible, we did not carry out the necessary computations.

(Equation (6) is their Eq. (3.8); the conjecture is stated in prose, not in a numbered environment.) Grading $\Lambda^*[\beta]$ by $\deg\beta = 1$, $\deg\mathsf{p}_k = k$, and writing $\Lambda^*[\beta]_{\leq n}$ for the elements of degree at most n, they then ask (Question 3.7, p. 8, verbatim): "Is it true for all $k \in \mathbb{N}$ that $\sum_{i=1}^{|\lambda|}\big(z_i^{(\beta)}(\lambda)\big)^k \in \Lambda^*[\beta]_{\leq 2k+2}$?" The introduction of [1] lists the same question among the open problems in the form "Does the Newton polynomial of degree k evaluated at the poles of monster potentials belong to the algebra of shifted symmetric functions on partitions?". The second power sum reappears in their Section 7: Corollary 7.7 restates (5) in terms of the eigenvalues $\mathcal{I}_1, \mathcal{I}_3$ of the quantum KdV hamiltonians diagonalised in their Theorem 7.6, and Remark 7.8 gives the analogous restatement of (6) "Conditional on the validity of the conjectural formula (3.8)".

The quantity itself is older than the source. Bazhanov, Lukyanov and Zamolodchikov [2] write $\sum_k z_k^2$ only as an unknown inside their expression for the third local integral of motion I_5, introduced by the sentence "We have checked by explicit diagonalization at the levels $L = 1, 2, 3$ that the eigenvalues of the first three local IM agree with the following expressions"; their I_5 is an explicit function of the level, the conformal dimension and the central charge plus multiples of $\sum_k z_k$ and of $\sum_k z_k^2$, and neither of the two sums is evaluated there (already their I_3 carries an unevaluated $\sum_k z_k$). The first power sum has an undeformed Hermite ancestor in Bonneux, Dunning and Stevens [3, Proposition 3 of the arXiv version v2]: the subleading coefficient of their R_λ is $r_{\lambda,1} = -c(\lambda)$ with $c(\lambda) = \sum_{(i,j)\in\lambda}(j-i)$ the sum of contents, which is $-\frac{1}{2}\mathsf{p}_3(\lambda)$ in the notation (4). And the second moment computed by Bonneux, Hamaker, Stembridge and Stevens [4] is a Plancherel average of the value of a Wronskian Appell polynomial, not a power sum of the zeros of one of them. Neither [3] nor [5], the two nearest studies of Wronskians of Hermite and of Laguerre eigenfunctions, contains the phrase "power sum" at all. We found no formula for $\sum_i (z_i^{(\beta)})^k$, $k \geq 2$, anywhere.

1.2. **What is settled here.** Everything below concerns the symmetric Wronskian $\Phi_{\lambda,\lambda'}$ of (3). Write $\Sigma_k(\lambda,\beta) = \sum_{i=1}^{|\lambda|}(z_i^{(\beta)}(\lambda))^k$.

Theorem 1.1 (the conjecture; proved in Section 5, not machine-checked as a whole)**.** *For every partition λ and every β, identity* (6) *holds.*

Theorem 1.2 (membership half of Question 3.7; proved in Section 5, not machine-checked as a whole)**.** *For every $k \geq 1$ there is an element $E_k \in \Lambda^*[\beta]_{\leq 8k}$ with $\Sigma_k(\lambda,\beta) = E_k(\lambda,\beta)$ for every partition λ and every β.*

Theorem 1.2 answers the question of the source's introduction affirmatively; what remains open of Question 3.7 is the sharp degree bound $2k+2$, which we prove for $k=2$ (Theorem 1.1: the right-hand side of (6) has degree 6) and verify numerically for $k=3$:

Theorem 1.3 (the third power sum; Theorem 6.1)**.** *There is an explicit $E \in \Lambda^*[\beta]_{\leq 8}$, displayed in (22), with $\Sigma_3(\lambda,\beta) = E(\lambda,\beta)$ as an identity of polynomials in β for every partition with $|\lambda| \leq 14$ (machine-checked at seven values of β, which decide it) and, by exact rational arithmetic outside the formal development, for every partition with $|\lambda| \leq 16$.*

The proof of Theorem 1.1 is a finite-determination argument. Its structural step (Lemma 5.1) is that, after the Wronskian matrix is reversed, the entry in row k and column i depends on the column only through one number, the *node* X_i ($= 2m_i + 2\beta$ for a ψ_+ column, $2n_j$ for a ψ_- column), because the p-th subleading Laguerre coefficient is the same polynomial $G_p(X) = (-1)^p\,(X/2)^{\underline{p}}\,(X/2-\beta)^{\underline{p}}/p!$ of the node for both kinds of column. The determinant is therefore alternating in the nodes, the Vandermonde divides it, and the coefficient $\widetilde{\phi}^{[d-k]}$ of $\widetilde{\Phi}$ is a symmetric polynomial in the nodes and β of total degree at most $2k$. Specialising the nodes to $\lambda+\delta_r$, $\lambda'+\delta_s$ (Lemma 5.2) and removing r,s with the source's Theorem 2.5 puts $\Sigma_2 - \mathrm{RHS}(6)$ inside $\Lambda^*[\beta]_{\leq 16}$. Each coefficient of a power of β then lies in the 231-dimensional space $\Lambda^*_{\leq 16}$, which is pinned down by the partitions of size at most 16 (Theorem 5.4, a rank certificate over $\mathbb{F}_{2^{31}-1}$, sharp at 509 columns), and (6) holds on all of them as an identity of polynomials in β (Propositions 4.1, 4.2 and Theorem 4.3). The route never computes $\widetilde{\phi}^{[d-4]}$ in closed form — the "necessary computations" the source declined — it only uses that $\widetilde{\phi}^{[d-4]}$ *is* a symmetric polynomial of bounded degree.

Around the proof we record what was verified exactly: (5) and (6) as identities of polynomials in β for every $|\lambda| \leq 17$, past the source's 15 (Theorem 4.3); both at five half-integer values of β, which by the source's Theorem 2.2 decide them, for $18 \leq |\lambda| \leq 21$ (Theorem 4.4); the equivalence of the source's two printed forms of each identity, Eq. (3.5) with Corollary 7.7 and Eq. (3.8) with Remark 7.8, as polynomial identities (Theorem 4.5); and the negative controls (Proposition 4.6). Finally, Observation 6.2 notes that the top-degree layer of Σ_k is row $2k$ of OEIS A088617 reversed for $k = 0,1,2,3$ — the $k=3$ layer was predicted from the triangle before the fit of Theorem 6.1 was run, and all seven predicted coefficients came out right.

2. The objects, as the source defines them

Throughout, $\mathbb{P}$ is the set of partitions, λ' the conjugate of λ, $\ell(\lambda)$ its length, $x^{\underline{p}} = x(x-1)\cdots(x-p+1)$ the falling factorial, $(x)_p = x(x+1)\cdots(x+p-1)$ the rising one, and $V(y_1,\dots,y_N) = \prod_{i<j}(y_j - y_i)$ the Vandermonde determinant, which the source writes Δ.

2.1. **Laguerre Wronskians.** By [1, Eq. (2.1)] the monic Laguerre polynomial has coefficients

$$c_n^{[i]}(\beta) = [y^i]\,\widehat{L}_n^{(\beta)}(y) = (-1)^{n+i}\binom{n}{i}(\beta+i+1)_{n-i} \qquad (0 \leq i \leq n). \tag{7}$$

For $r,s \in \mathbb{N}$ let $\mathcal{V}_r$ be the set of strictly decreasing tuples $\boldsymbol{m} = (m_1 > \cdots > m_r)$ of natural numbers. For $\boldsymbol{m} \in \mathcal{V}_r$, $\boldsymbol{n} \in \mathcal{V}_s$ the source defines (Eq. (2.7)) the Wronskian (2) and (Eq. (2.8))

$$\kappa^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}} = V(2\boldsymbol{m}+\beta,\, 2\boldsymbol{n}-\beta), \tag{8}$$

the Vandermonde of the $N = r+s$ numbers $2m_1+\beta,\dots,2m_r+\beta, 2n_1-\beta,\dots,2n_s-\beta$. We use the following parts of [1, Theorem 2.2]: for all r,s, $\boldsymbol{m} \in \mathcal{V}_r$, $\boldsymbol{n} \in \mathcal{V}_s$,

$$\Psi^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(x) = \kappa^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}\, e^{-\frac{1}{2}(r+s)x^2}\, x^{\frac{1}{2}(r-s)^2+\beta(r-s)}\, \widetilde{\Phi}^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(x^2), \tag{9}$$

where $\widetilde{\Phi}^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(y) \in \mathbb{Z}[\beta,y]$ is monic in y of degree

$$d = |\boldsymbol{m}| + |\boldsymbol{n}| - \tbinom{r}{2} - \tbinom{s}{2}, \tag{10}$$

and, writing $\widetilde{\Phi}^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(y)=\sum_{k=0}^{d}\widetilde{\phi}^{[k]}_{\boldsymbol{m},\boldsymbol{n}}(\beta)\,y^k$ (Eq. (2.11)), each $\widetilde{\phi}^{[k]}_{\boldsymbol{m},\boldsymbol{n}}(\beta)\in\mathbb{Z}[\beta]$ has degree at most $d-k$ in β. The theorem also gives $\widetilde{\phi}^{[d-1]}$ and $\widetilde{\phi}^{[d-2]}$ in closed form (Eqs. (2.13) and (2.14)) in terms of the power sums $\rho^{\pm}_k=\sum_i m_i^k\pm\sum_j n_j^k$; we use those two formulas only as controls (Remark 3.1).

2.2. **Partitions and the symmetric case.** With $\delta_r=(r-1,r-2,\dots,0)$ and $\lambda+\delta_r\in\mathcal{V}_r$ for $r\ge\ell(\lambda)$, [1, Theorem 2.5] states that $\widetilde{\Phi}^{(\beta-r+s)}_{\lambda+\delta_r,\mu+\delta_s}$ depends only on β,λ,μ and not on $r\ge\ell(\lambda)$, $s\ge\ell(\mu)$, and [1, Definition 2.6, Eq. (2.18)] sets

$$\Phi^{(\beta)}_{\lambda,\mu}(y)=\widetilde{\Phi}^{(\beta-r+s)}_{\lambda+\delta_r,\mu+\delta_s}(y), \tag{11}$$

a polynomial of degree $|\lambda|+|\mu|$. [1, Theorem 3.1] gives $\Phi^{(\beta)}_{\lambda',\mu'}(y)=(-1)^{|\lambda|+|\mu|}\Phi^{(\beta)}_{\mu,\lambda}(-y)$, so $\Phi^{(\beta)}_{\lambda,\lambda'}$ is even, and (3) defines $\mathcal{P}^{(\beta)}_{\lambda}$ and its zeros $z^{(\beta)}_i(\lambda)$ (Definition 3.4, Eqs. (3.2)–(3.3)). The shifted symmetric functions are (4) (Definition 3.5, Eq. (3.4)); they satisfy $\mathsf{p}_k(\lambda')=(-1)^k\mathsf{p}_k(\lambda)$, $\mathsf{p}_2(\lambda)=|\lambda|$, and $\Sigma_0=|\lambda|=\mathsf{p}_2$ is the "obvious identity" of the source. We write $\Lambda^*_{\le n}$ for the span of the monomials $\mathsf{p}_\mu=\prod_i\mathsf{p}_{\mu_i}$ over partitions μ with all parts ≥ 2 and $|\mu|\le n$ (the empty product 1 included); $\dim\Lambda^*_{\le n}=p(n)$, the number of partitions of n, and $\Lambda^*[\beta]_{\le n}=\bigoplus_{a=0}^{n}\beta^a\Lambda^*_{\le n-a}$.

2.3. **The quantum-KdV restatements.** [1, Theorem 7.6] diagonalises the first three quantum KdV hamiltonians at $c=-2$ on the Schur functions, with eigenvalues (Eq. (7.37))

$$\begin{aligned}\mathcal{I}_1&=\tfrac{\beta^2-1}{8}+\mathsf{p}_2,\qquad \mathcal{I}_3=\Big(\tfrac{\beta^2-1}{8}\Big)^2+\tfrac{3}{4}\beta^2\mathsf{p}_2+\tfrac{3}{2}\beta\mathsf{p}_3+\mathsf{p}_4,\\ \mathcal{I}_5&=\Big(\tfrac{\beta^2-1}{8}\Big)^3+\tfrac{15\beta^4}{64}\mathsf{p}_2+\tfrac{15\beta^3}{16}\mathsf{p}_3+\beta^2\Big(\tfrac{15}{8}\mathsf{p}_4+\tfrac{3}{16}\mathsf{p}_2\Big)+\beta\Big(\tfrac{15}{8}\mathsf{p}_5+\tfrac{3}{8}\mathsf{p}_3\Big)+\tfrac{3}{4}\mathsf{p}_6+\tfrac{\mathsf{p}_4}{4}.\end{aligned} \tag{12}$$

Corollary 7.7 (Eq. (7.38)) reads $\Sigma_1=2\mathcal{I}_3-2\mathcal{I}_1^2$, and Remark 7.8 (Eq. (7.39)), conditional on (6), reads $\Sigma_2=\frac{56}{3}\mathcal{I}_5-40\,\mathcal{I}_3\mathcal{I}_1+\frac{64}{3}\mathcal{I}_1^3+12\,\mathcal{I}_3-12\,\mathcal{I}_1^2$.

3. The reduction to a truncated determinant

Fix r,s, $\boldsymbol{m}\in\mathcal{V}_r$, $\boldsymbol{n}\in\mathcal{V}_s$ and β; put $t=2\beta$, $N=r+s$ and define the *nodes*

$$X_i=2m_i+t\quad(1\le i\le r),\qquad X_{r+j}=2n_j\quad(1\le j\le s). \tag{13}$$

Common factor. Since $\mathrm{Wr}(fg_1,\dots,fg_N)=f^N\,\mathrm{Wr}(g_1,\dots,g_N)$, taking $f=e^{-x^2/2}x^{1/2-\beta}$ leaves the columns $g_i=x^t\widehat{L}^{(\beta)}_{m_i}(x^2)$ and $g_{r+j}=\widehat{L}^{(-\beta)}_{n_j}(x^2)$.

Derivatives. Each column is $g=\sum_a c_a x^{2a+\tau}$ with $\tau=t$ or 0, so $\partial_x^k g=\sum_a c_a\,(2a+\tau)^{\underline{k}}x^{2a+\tau-k}$. Multiplying row k of the Wronskian matrix by x^k (which multiplies the determinant by $x^{\binom{N}{2}}$) makes the entry in row k, column i equal to $\sum_a c_a(2a+\tau)^{\underline{k}}x^{2a+\tau}$. For a column of Laguerre degree m and node X one has $2a+\tau=X-2p$ with $p=m-a$; factoring $x^\tau u^m$ out of the column, $u=x^2$, the entry becomes $x^\tau u^m\sum_{p\ge0}c^{[m-p]}\,(X-2p)^{\underline{k}}u^{-p}$.

Reversal. Collecting the factors $x^{rt}u^{|\boldsymbol{m}|+|\boldsymbol{n}|}$ and writing $v=1/u$ gives

$$\begin{aligned}\mathrm{Wr}(g_1,\dots,g_N)&=x^{rt-\binom{N}{2}}\,u^{|\boldsymbol{m}|+|\boldsymbol{n}|}\det\check{M}(1/u),\\ \check{M}_{k,i}(v)&=\sum_{p\ge0}G_p(X_i)\,(X_i-2p)^{\underline{k}}\,v^p\qquad(0\le k\le N-1,\ 1\le i\le N),\end{aligned} \tag{14}$$

where $G_p(X_i)=c^{[m_i-p]}_{m_i}(\beta)$ for a ψ_+ column and $G_p(X_{r+j})=c^{[n_j-p]}_{n_j}(-\beta)$ for a ψ_- column.

The coefficient function is the same for both kinds of column. By (7), $c^{[m-p]}_m(\beta)=(-1)^p\binom{m}{p}(\beta+m-p+1)_p=\frac{(-1)^p}{p!}\,m^{\underline{p}}\,(\beta+m)^{\underline{p}}$. For a ψ_+ column, $X/2=m+\beta$ and $X/2-\beta=m$; for a ψ_- column, with parameter $-\beta$ and $X=2n$, $X/2=n$ and $X/2-\beta=n-\beta$. In both cases

$$G_p(X)=\frac{(-1)^p}{p!}\,\Big(\frac{X}{2}\Big)^{\underline{p}}\,\Big(\frac{X}{2}-\beta\Big)^{\underline{p}}, \tag{15}$$

one and the same polynomial in X and β, of total degree $2p$. This is the observation everything below rests on. It also makes $G_p(X_i) = 0$ as soon as p exceeds the Laguerre degree of column i, so the sum in (14) is finite and any truncation $v^{K+1} = 0$ with K at least the order wanted is exact.

Identification with $\widetilde{\Phi}$. The v^0-coefficient of $\det \check{M}$ is $\det(X_i^{\underline{k}})_{k,i} = V(\boldsymbol{X})$, and $V(\boldsymbol{X}) = \kappa^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}$ by (8) and translation invariance of the Vandermonde. Comparing (14) with (9) (the exponents of x agree because $\frac{1}{2}(r-s)^2 = \frac{N}{2} - \binom{N}{2} + 2\binom{r}{2} + 2\binom{s}{2}$) gives $u^{|\boldsymbol{m}|+|\boldsymbol{n}|} \det \check{M}(1/u) = \kappa\, u^{\binom{r}{2}+\binom{s}{2}}\, \widetilde{\Phi}^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}}(u)$, that is,

$$\det \check{M}(v) = \kappa^{(\beta)}_{\boldsymbol{m},\boldsymbol{n}} \sum_{k=0}^{d} \widetilde{\phi}^{[d-k]}_{\boldsymbol{m},\boldsymbol{n}}(\beta)\, v^k. \tag{16}$$

Vieta. In the symmetric case $\boldsymbol{m} = \lambda + \delta_r$, $\boldsymbol{n} = \lambda' + \delta_s$, parameter $\beta - r + s$, we have $d = 2|\lambda|$ and $\mathcal{P}^{(\beta)}_\lambda(z) = \sum_j \widetilde{\phi}^{[2j]} z^j$ by (3), so with $e_1 = -\widetilde{\phi}^{[d-2]}$, $e_2 = \widetilde{\phi}^{[d-4]}$, $e_3 = -\widetilde{\phi}^{[d-6]}$ the elementary symmetric functions of the zeros,

$$\Sigma_1 = e_1, \qquad \Sigma_2 = e_1^2 - 2e_2, \qquad \Sigma_3 = e_1^3 - 3e_1e_2 + 3e_3. \tag{17}$$

So Σ_1, Σ_2 need only the first five coefficients of $\det \check{M}$, i.e. arithmetic in $\mathbb{Z}[v]/(v^5)$, and Σ_3 needs $\mathbb{Z}[v]/(v^7)$.

Degree in β. Expanding $\det \check{M}$ by multilinearity in the columns,

$$[v^k] \det \check{M} = \sum_{p_1+\cdots+p_N=k} \prod_{i=1}^{N} G_{p_i}(X_i) \cdot V(X_1 - 2p_1, \ldots, X_N - 2p_N). \tag{18}$$

At the nodes (13), $G_p(X_i)$ has degree p in β (through $(m_i+\beta)^{\underline{p}}$ for a ψ_+ node, $(n_j-\beta)^{\underline{p}}$ for a ψ_- node), and the Vandermonde has degree at most rs (only a ψ_+/ψ_- pair of nodes differs by a multiple of β). Hence, writing $\phi_k = [v^k] \det \check{M}$ as a function of β,

$$\deg_\beta \phi_k \le rs+k, \qquad \deg_\beta\big(\phi_2^2 - 2\phi_4\phi_0 - R(\beta)\phi_0^2\big) \le 2rs+4, \qquad \deg_\beta\big(-8\phi_2 - 8R_1(\beta)\phi_0\big) \le rs+2, \tag{19}$$

where R and R_1 are the right-hand sides of (6) and (5). Vanishing at $2rs+5$ distinct values of β therefore forces the polynomial identity behind (6), and vanishing at $rs+3$ values the one behind (5). This bound is proved here and does not use the β-degree statement of Theorem 2.2; the statement of Theorem 2.2, $\deg_\beta \widetilde{\phi}^{[d-k]} \le k$, gives the weaker but simpler fact that both sides of (6) have β-degree at most 4, so that *five* values of β decide (6) and (5) at a given λ, and that both sides of the $k = 3$ identity (22) have β-degree at most 6, so that seven values decide it.

Exact evaluation. All computations below evaluate at half-integers $\beta = T/2$, T odd. Then every ψ_+ node is an odd integer and every ψ_- node an even integer, so the nodes are distinct and $\kappa \neq 0$ automatically (κ vanishes only at integer β). Every entry is multiplied by $6144 = 4^4 \cdot 4!$ (resp. $2949120 = 4^6 \cdot 6!$ for the v^7 truncation), which makes G_p integral for $p \le 4$ (resp. $p \le 6$); a uniform scaling of the matrix multiplies every ϕ_k by 6144^N and leaves the ratios ϕ_k/ϕ_0 untouched. The falling factorials $(X_i - 2p)^{\underline{k}}$ in row k are replaced by the monomials $(X_i - 2p - c)^k$ with a common translation c (a unitriangular change of row basis, which leaves the determinant unchanged), and the determinant over $\mathbb{Z}[v]/(v^5)$ is computed by fraction-free (Bareiss) elimination, pivoting on constant terms; exact division in the truncated ring recovers the truncation of the true integral quotient coefficient by coefficient. With $P_k = 2^{k-1}\mathsf{p}_k(\lambda)$ (an integer) and $T = 2\beta$, the identities checked are, in cleared form,

$$\begin{gathered} -8\phi_2 = \mathrm{propNum} \cdot \phi_0, \qquad 96\big(\phi_2^2 - 2\phi_4\phi_0\big) = \mathrm{rhsNum} \cdot \phi_0^2, \\ 128\big(-\phi_2^3 + 3\phi_2\phi_4\phi_0 - 3\phi_6\phi_0^2\big) = \mathrm{cubeNum} \cdot \phi_0^3, \end{gathered} \tag{20}$$

where $\mathrm{propNum} = 8R_1$, $\mathrm{rhsNum} = 96R$ and $\mathrm{cubeNum} = 128{\cdot}\mathrm{RHS}(22)$ are the right-hand sides written as integer polynomials in $P_2, \ldots, P_8, T$ (Appendix A).

Controls at every evaluation. Two independent facts are asserted at every single (λ, β) visited: $\phi_0 = 6144^N \prod_{i<j}(X_j - X_i)$, i.e. the constant term returned by the elimination equals the independently computed Vandermonde, the source's κ of (8); and the first identity of (20), i.e. the source's *proved* Proposition 3.6. A slip in the reduction or in the exact division cannot pass either.

Remark 3.1 (the source's Theorem 2.2 as a further control). The closed forms [1, Eqs. (2.13), (2.14)] for $\widetilde{\phi}^{[d-1]}$ and $\widetilde{\phi}^{[d-2]}$ in $\rho_k^{\pm}$, N, r, s were transcribed and compared, in exact rational arithmetic outside the formal development, with ϕ_1/ϕ_0 and ϕ_2/ϕ_0 from (14) on 26 random $(\boldsymbol{m}, \boldsymbol{n}, \beta)$ with $1 \le r, s \le 4$, distinct $m_i, n_j \in [0, 12)$ and half-integer β avoiding the zeros of κ: no mismatch for either formula. Specialising (18) at $k = 1$ and eliminating ρ_1^- from Eq. (2.13) gives, for arbitrary nodes and β, $\widetilde{\phi}^{[d-1]} = -\frac{1}{4}P_2 + \frac{1}{2}(\beta + N - 1)P_1 - \frac{1}{2}\beta N(N-1) - \binom{N}{3}$ with $P_a = \sum_i X_i^a$; this was checked on 40 random $(N, \boldsymbol{X}, \beta)$, $2 \le N \le 7$, integer nodes in $[-30, 30)$, half-integer β, again with no mismatch. It confirms at once the identity (15), the symmetry of Lemma 5.1 and the transcription of Eq. (2.13).

4. Machine-checked identities

Each statement of this section was verified inside the formal development described in Section 7; the Lean statements are reproduced verbatim in Appendix A. In all of them "for every partition of n" means for every element of the list of weakly decreasing sequences produced by the development's own generator, whose length is checked against $p(n)$ for $n \le 12$ in Proposition 4.6(1), and "as an identity of polynomials in β" means verified at the $2rs + 5$ points $\beta = \frac{1}{2}, \frac{3}{2}, \ldots$, $r = \ell(\lambda)$, $s = \lambda_1$, which by (19) is equivalent.

4.1. **The source's range.**

Proposition 4.1 (Proposition 3.6 re-derived)**.** *For every partition λ with $1 \le |\lambda| \le 16$, identity* (5) *holds as an identity of polynomials in β, and at every one of the* 54,684 *evaluations (λ, β) involved the constant term ϕ_0 of the eliminated determinant equals $6144^N \prod_{i<j}(X_j - X_i)$, the source's κ.*

Proposition 4.2 (Eq. (3.8) on the source's range, exactly and for all β)**.** *For every partition λ with $1 \le |\lambda| \le 15$, identity* (6) *holds as an identity of polynomials in β.*

Propositions 4.1 and 4.2 are the reproduction step: from the source's definitions alone — the development holds no data, it generates the partitions, computes the Laguerre coefficients from (7), builds the matrix and takes the determinant — its *proved* Proposition 3.6 is recovered at every evaluation, and its conjecture is recovered on exactly the range it reports, as a polynomial identity rather than at sampled values. The right-hand sides of (5) and (6) are the only numbers typed in.

4.2. **Past the source's range.**

Theorem 4.3 (Eq. (3.8) for $|\lambda| = 16, 17$)**.** *For every partition λ of* 16 *and every partition of* 17 *(*$231 + 297$ *partitions,* $17{,}351 + 24{,}281 = 41{,}632$ *determinant evaluations), identities* (5) *and* (6) *hold as identities of polynomials in β.*

Theorem 4.4 (five values of β, $18 \le |\lambda| \le 21$)**.** *For every partition λ with $18 \le |\lambda| \le 21$ (*$385 + 490 + 627 + 792 = 2{,}294$ *partitions) and every $\beta \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}$, the Vandermonde control and the cleared identities* (20) *for* (5) *and* (6) *hold. Consequently, by the β-degree bound of* [1, Theorem 2.2]*, identities* (5) *and* (6) *hold at those λ for every β.*

The polynomial-identity check costs $2rs + 5$ determinants per partition and grows by a factor of about 1.75 per unit of $|\lambda|$; Table 1 records the sweep. The five-point check is about eleven times cheaper but leans on the source's Theorem 2.2 for the step from five values of β to all of them, where Propositions 4.1, 4.2 and Theorem 4.3 do not.

4.3. **Transcription: the two printed forms of each identity.**

Theorem 4.5 (Eq. (3.5) = Corollary 7.7, Eq. (3.8) = Remark 7.8)**.** *As identities in $\mathbb{Q}[\beta, \mathsf{p}_2, \ldots, \mathsf{p}_6]$, with $\mathcal{I}_1, \mathcal{I}_3, \mathcal{I}_5$ the polynomials* (12)*:*

$$2\mathsf{p}_4 - 2\mathsf{p}_2^2 + 3\beta\mathsf{p}_3 + \beta^2\mathsf{p}_2 + \tfrac{1}{2}\mathsf{p}_2 = 2\mathcal{I}_3 - 2\mathcal{I}_1^2,$$

$$\textit{RHS of } (6) = \tfrac{56}{3}\mathcal{I}_5 - 40\,\mathcal{I}_3\mathcal{I}_1 + \tfrac{64}{3}\mathcal{I}_1^3 + 12\,\mathcal{I}_3 - 12\,\mathcal{I}_1^2.$$

$\lvert\lambda\rvert$	partitions	determinant evaluations	where checked
1–9	96	2,652	Lean and exact Python
10, 11, 12, 13	42, 56, 77, 101	1,704; 2,588; 3,943; 5,789	Lean and exact Python
14, 15	135, 176	8,491; 12,166	Lean and exact Python
16, 17	231, 297	17,351; 24,281	Lean and exact Python
18	385	33,875	exact Python only

TABLE 1. The all-β sweep: identities (5) and (6) at $2rs+5$ half-integer points per partition, no failure at any size. Exact rational arithmetic outside the formal development also confirmed the five-point check of Theorem 4.4 for every partition with $1 \le |\lambda| \le 25$ (9,295 partitions).

Proof. Expand and compare coefficients; the formal proof is the tactic `ring` over $\mathbb{Q}$ and is the only part of the development that needs no evaluation by compiled code. Its value is as a transcription control: every coefficient of the four displayed formulas (5), (6), (12) enters, and a single misread digit breaks it. In particular Theorem 1.1 makes the source's Remark 7.8 unconditional. □

4.4. **Controls.**

Proposition 4.6 (negative controls and non-vacuity)**.** (1) *The partition generator lists* 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77 *partitions of* $n = 0, 1, \dots, 12$*, the values of* $p(n)$*.*

(2) *Every partition of* 6 *is tested at* $2rs + 5 \ge 7$ *values of* β*.*

(3) *The data is not identically zero: for* $\lambda = (3, 2, 1)$*,* $(P_2, P_3, P_4, P_5, P_6) = (12, 0, 252, 0, 6252)$ *with* $P_k = 2^{k-1}\mathsf{p}_k$*; here* $P_2 = 2|\lambda|$ *and the odd ones vanish because* $(3, 2, 1)$ *is self-conjugate.*

(4) *(Too small a claim.) On the partitions of* 4*, at* $\beta = \frac{1}{2}, \dots, \frac{9}{2}$*, the cleared form of* (6) *holds, while each of the three mutilated right-hand sides — the* $\beta^4\mathsf{p}_2$ *term deleted, the* $\frac{27}{8}\mathsf{p}_2$ *term deleted,* $14\mathsf{p}_6$ *changed to* $15\mathsf{p}_6$ *— fails.*

(5) *(Too large a claim.) Identity* (6) *is about the symmetric Wronskian* $\Phi_{\lambda,\lambda'}$*. At* $\lambda = (3, 1)$ *and* $\beta = \frac{1}{2}, \dots, \frac{9}{2}$*, the quantity* $(\widetilde{\phi}^{[d-2]})^2 - 2\widetilde{\phi}^{[d-4]}$ *built from* $\Phi_{\lambda,\mu}$ *via* (11) *equals the right-hand side of* (6) *for* $\mu = \lambda' = (2, 1, 1)$ *and differs from it for each of the other four partitions* μ *of* 4*.*

(6) *(Sharpness of Theorem 5.4.) The evaluation matrix of Theorem 5.4 restricted to its first* 508 *columns has rank* 230 *over* $\mathbb{F}_{2^{31}-1}$*.*

5. The proof of the conjecture

Throughout this section $\check{M}$ is the matrix (14) with the entries (15), regarded as a matrix over the polynomial ring $\mathbb{Q}[\beta][X_1, \dots, X_N]$ with a formal variable v truncated at any order $K \ge k$ (the entry in row k, level p, is $G_p(X_i)(X_i - 2p)^{\underline{k}}$); nothing about the nodes being of the form (13) is used until Lemma 5.2.

Lemma 5.1 (symmetry and degree)**.** *For every* $k \ge 0$*, the polynomial* $[v^k] \det \check{M}$ *is divisible by* $V(\boldsymbol{X})$ *in the ring* $\mathbb{Q}[\beta][X_1, \dots, X_N]$*, and the quotient*

$$S_k(\boldsymbol{X}; \beta) = \frac{[v^k] \det \check{M}}{V(\boldsymbol{X})}$$

is a symmetric polynomial in $X_1, \dots, X_N$ *with coefficients in* $\mathbb{Q}[\beta]$*, of total degree at most* $2k$ *in* $(X_1, \dots, X_N, \beta)$ *jointly. At the nodes* (13)*,* $S_k(\boldsymbol{X}; \beta) = \widetilde{\phi}^{[d-k]}_{\boldsymbol{m},\boldsymbol{n}}(\beta)$*.*

Proof. By (15) the entry $\check{M}_{k,i}$ depends on the column index i only through X_i. Hence every v-coefficient of $\det \check{M}$ is an alternating polynomial in $X_1, \dots, X_N$ over $\mathbb{Q}[\beta]$: transposing two columns changes its sign, and it vanishes when two nodes coincide. It is therefore divisible by each $X_j - X_i$, $i < j$, hence by their product $V(\boldsymbol{X})$ (these are pairwise coprime in the factorial ring $\mathbb{Q}[\beta][\boldsymbol{X}]$), and the quotient is symmetric. Degree: the entry in row k at level p has degree $k + 2p$ in $(\boldsymbol{X}, \beta)$; a term of the Leibniz expansion contributing to v^k uses each row once and levels p_i with $\sum_i p_i = k$, so it has degree at most

$\sum_{j=0}^{N-1} j + 2k = \binom{N}{2} + 2k$, and $\deg V = \binom{N}{2}$. The last assertion is (16) together with $V(\boldsymbol{X}) = \kappa \neq 0$ as a polynomial in β. □

Lemma 5.2 (specialisation to partitions)**.** *Let $\lambda, \mu \in \mathbb{P}$, $r \geq \ell(\lambda)$, $s \geq \ell(\mu)$, $\boldsymbol{m} = \lambda + \delta_r$, $\boldsymbol{n} = \mu + \delta_s$, and let the nodes be* (13) *with parameter $\beta_{\mathrm{in}} = \beta - r + s$ in place of β. Then for every $a \geq 1$ the node power sum $P_a = \sum_{i=1}^{N} X_i^a$ is a polynomial in β, r, s, $\mathsf{p}_2(\lambda), \dots, \mathsf{p}_{a+1}(\lambda)$ and $\mathsf{p}_2(\mu), \dots, \mathsf{p}_{a+1}(\mu)$, of weight at most $a+1$ for the weights $\operatorname{wt}\beta = 1$, $\operatorname{wt}\mathsf{p}_k = k$, $\operatorname{wt} r = \operatorname{wt} s = 0$.*

Proof. For $i \leq r$, $X_i = 2(\lambda_i + r - i) + 2\beta_{\mathrm{in}} = 2\big(x_i + r - \frac{1}{2} + \beta_{\mathrm{in}}\big)$ with $x_i = \lambda_i - i + \frac{1}{2}$, and for every $b \geq 0$, $\sum_{i=1}^{r} x_i^b = \mathsf{p}_{b+1}(\lambda) + \sum_{i=1}^{r}(-i+\frac{1}{2})^b$ by (4), because the summands of (4) vanish for $i > \ell(\lambda)$ and $r \geq \ell(\lambda)$; the last sum is an explicit polynomial in r. Expanding $(x_i + r - \frac{1}{2} + \beta_{\mathrm{in}})^a$ binomially gives terms $x_i^b \beta^c$ with $b + c \leq a$, of weight $(b+1) + c \leq a+1$ after summation. The ψ_- nodes $X_{r+j} = 2(\mu_j + s - j)$ are the same with μ, s and no β. □

The source's Eq. (3.6) is the same computation in the cases $a = 1, 2, 3$, written for the power sums $\rho_a^{\pm}$ of $\boldsymbol{m}, \boldsymbol{n}$ rather than for the nodes and specialised to $\mu = \lambda'$.

Lemma 5.3 (weight of the power sums, r, s fixed)**.** *Fix $k \geq 1$ and $M \geq 1$. There is a polynomial $E_{k,M} \in \Lambda^*[\beta]_{\leq 8k}$ such that $\Sigma_k(\lambda, \beta) = E_{k,M}(\lambda, \beta)$ for every partition λ with $\ell(\lambda) \leq M$ and $\lambda_1 \leq M$ and every β.*

Proof. Take $r = s = M$, so $N = 2M$ is fixed, and use (11): for λ in the $M \times M$ box, $\Phi^{(\beta)}_{\lambda,\lambda'} = \widetilde{\Phi}^{(\beta)}_{\lambda+\delta_M, \lambda'+\delta_M}$ (the shift $-r + s$ is zero). By Newton's identities Σ_k is a polynomial with rational coefficients in $e_1, \dots, e_k$, isobaric of weight k when e_j has weight j, and $e_j = (-1)^j \widetilde{\phi}^{[d-2j]} = (-1)^j S_{2j}(\boldsymbol{X}; \beta)$ by (17) and Lemma 5.1. By Lemma 5.1, S_{2j} is a symmetric polynomial in the nodes and β of total degree at most $4j$, hence (fundamental theorem of symmetric polynomials, in the power-sum basis over $\mathbb{Q}$) a polynomial in $\beta, P_1, P_2, \dots$ in which every monomial $\beta^a \prod_l P_{a_l}$ has $a + \sum_l a_l \leq 4j$, so at most $4j - a$ factors P_{a_l}. By Lemma 5.2 such a monomial specialises to a polynomial in β, M and the $\mathsf{p}_i(\lambda)$, $\mathsf{p}_i(\lambda') = (-1)^i \mathsf{p}_i(\lambda)$, of weight at most $a + \sum_l (a_l + 1) \leq 4j + (4j - a) \leq 8j$. So each e_j has weight at most $8j$, and Σ_k, isobaric of weight k in the e_j, has weight at most $8k$; with M fixed this is an element of $\Lambda^*[\beta]_{\leq 8k}$. □

Theorem 5.4 (the finite-determination certificate)**.** *There are $231 = p(16)$ partitions μ with all parts ≥ 2 and $|\mu| \leq 16$ (the empty partition included), and 915 partitions of size at most 16. Let $\mathcal{S}$ be the set consisting of the 508 partitions of size at most 14 (the empty partition included) together with the partition (1^{15}); these are the first 509 partitions of size at most 16 in the development's order. The 231×509 matrix with entries $\prod_i P_{\mu_i}(\lambda)$, $P_k = 2^{k-1}\mathsf{p}_k$, reduced modulo the prime $2^{31} - 1$, has rank 231 over $\mathbb{F}_{2^{31}-1}$. Consequently the 231 monomials p_μ are linearly independent as functions on $\mathcal{S}$, and an element of $\Lambda^*_{\leq 16}$ vanishing on every partition of size at most 16 (indeed, on $\mathcal{S}$) is zero.*

Proof. The rank is computed by Gaussian elimination over $\mathbb{F}_{2^{31}-1}$ inside the formal development. The rank of an integer matrix over $\mathbb{F}_{2^{31}-1}$ is at most its rank over $\mathbb{Q}$, so the 231 rows are independent over $\mathbb{Q}$; scaling row μ by the nonzero constant $2^{|\mu| - \ell(\mu)}$ does not change this. Proposition 4.6(6) shows 509 is the least number of columns that works: on the 508 partitions of size at most 14 the rank is 230, and column 509 is the first partition of 15. That is the expected threshold. Okounkov and Olshanski filter Λ^* by degree, with $\operatorname{gr}\Lambda^* \cong \Lambda$ [7, Definition 1.1 and Proposition 1.5]; the shifted Schur functions s^*_ν, $\nu \in \mathbb{P}$, form a linear basis of Λ^* [7, Corollary 1.6(1)], with $\deg s^*_\nu = |\nu|$ and top term the ordinary Schur function s_ν [7, Theorem 3.4]; and the Vanishing Theorem [7, Theorem 3.1] says

$$s^*_\nu(\lambda) = 0 \ \text{ unless } \ \nu \subseteq \lambda, \qquad s^*_\nu(\nu) = H(\nu) \neq 0, \tag{21}$$

$H(\nu)$ the product of the hook lengths of ν. Our p_k is the source's normalisation, a shifted power sum with exponent $k-1$ and shift $\frac{1}{2}$; it is *not* the p^*_k of [7], whose exponent is k and whose degree is k. Since the shift is immaterial (Definition 1.1) and the top term of p_k is the ordinary power sum p_{k-1}, one has $\deg \mathsf{p}_k = k - 1$ and hence $\deg \mathsf{p}_\mu = |\mu| - \ell(\mu)$, which over the 231 monomials reaches 15, at $\mu = (16)$. So $\Lambda^*_{\leq 16}$ sits inside the span of the s^*_ν with $|\nu| \leq 15$, and (21) shows that partitions of size up to 15 are

exactly what is needed: the certificate could be replaced by that citation, and the rank computation is used so that the step is verified here instead. □

Proof of Theorem 1.2. Fix k and let $M' \geq M \geq \max(16, 8k)$. By Lemma 5.3 and the source's Theorem 2.5 (which says that $\Sigma_k(\lambda, \beta)$ does not depend on the box chosen), $E_{k,M}$ and $E_{k,M'}$ agree on every λ in the $M \times M$ box, in particular on every partition with $|\lambda| \leq M$, for every β. Write $E_{k,M} - E_{k,M'} = \sum_a \beta^a D_a$ with $D_a \in \Lambda^*_{\leq 8k-a}$; for each such λ the polynomial in β vanishes identically, so every $D_a(\lambda) = 0$. For $k \leq 2$, $D_a \in \Lambda^*_{\leq 16}$ vanishes on all partitions of size at most 16, hence $D_a = 0$ by Theorem 5.4. For general k, $D_a \in \Lambda^*_{\leq 8k}$ lies in the span of the shifted Schur functions s^*_ν with $|\nu| \leq 8k-1$ (the degree count in the proof of Theorem 5.4, and [7, Corollary 1.6(1)] for the basis), and it vanishes on all partitions of size at most $8k-1 \leq M$. The span is finite, so if $D_a = \sum_\nu c_\nu s^*_\nu \neq 0$ we may take ν_0 minimal for inclusion among the finitely many ν with $c_\nu \neq 0$: by (21), $s^*_\nu(\nu_0) = 0$ unless $\nu \subseteq \nu_0$, and minimality leaves only $\nu = \nu_0$, so $D_a(\nu_0) = c_{\nu_0} H(\nu_0) \neq 0$ — a contradiction, since $|\nu_0| \leq 8k-1$. Hence $E_{k,M} = E_{k,M'} =: E_k$ for all $M' \geq M$, and since every partition lies in some box, $\Sigma_k = E_k$ everywhere. For $k \leq 2$ the proof uses only Theorem 5.4, not the cited vanishing theorem. □

For $k = 1$ this gives weight at most 8 where Proposition 3.6 has weight 4; for $k = 2$ it gives $\Lambda^*[\beta]_{\leq 16}$ where Question 3.7 asks for 6. The crude factor 4 comes from the last two lines of the proof of Lemma 5.3, and a sharper degree bound in Lemma 5.1 is the lever for the remaining half of the question (Remark 6.3).

Proof of Theorem 1.1. Let R be the right-hand side of (6) and $D = E_2 - R$ with E_2 from Theorem 1.2. Inspecting R, every monomial has weight 6, 4 or 2, so $R \in \Lambda^*[\beta]_{\leq 6} \subseteq \Lambda^*[\beta]_{\leq 16}$ and $D \in \Lambda^*[\beta]_{\leq 16}$; write $D = \sum_{a=0}^{16} \beta^a D_a$ with $D_a \in \Lambda^*_{\leq 16-a} \subseteq \Lambda^*_{\leq 16}$. By Propositions 4.1, 4.2 and Theorem 4.3, $\Sigma_2(\lambda, \beta) = R(\lambda, \beta)$ for every partition with $1 \leq |\lambda| \leq 16$ as an identity of polynomials in β (the cleared identity (20) holds at $2rs + 5$ points, which by (19) is the polynomial identity, and $\phi_0 = \kappa \not\equiv 0$); at $\lambda = \varnothing$ both sides of (6) are 0, so the identity holds on all of $|\lambda| \leq 16$. So for each such λ the polynomial $D(\lambda, \cdot)$ vanishes, hence every $D_a(\lambda) = 0$; each D_a then vanishes on all partitions of size at most 16, so $D_a = 0$ by Theorem 5.4. Thus $D = 0$, i.e. $\Sigma_2 = R$ for every λ and every β. □

Remark 5.5 (what the proof rests on). From the source: Theorem 2.2, used only to know that Ψ has the form (9) with $\widetilde{\Phi}$ monic of degree (10) and κ as in (8) (the Vandermonde control re-derives κ at every evaluation), Theorem 3.1 (evenness, so that (3) and (17) make sense), and Theorem 2.5 (independence of r, s). New and elementary: (15) and Lemmas 5.1–5.3, which are proved on paper only. Finite and machine-checked: the polynomial identities for $|\lambda| \leq 16$ (Propositions 4.1, 4.2, Theorem 4.3) and the certificate (Theorem 5.4). The β-degree bound of Theorem 2.2 and the shifted Schur vanishing theorem (21) are *not* used in Theorem 1.1. The source's suggested route, "the methods used in Proposition 3.6", would specialise Theorem 2.2, whose closed forms are the three leading coefficients $\widetilde{\phi}^{[d]}$, $\widetilde{\phi}^{[d-1]}$, $\widetilde{\phi}^{[d-2]}$ and the constant term $\widetilde{\phi}^{[0]}$, whereas Σ_2 needs $\widetilde{\phi}^{[d-4]}$ — two more orders of an already half-page formula.

Remark 5.6 (an empirical control on Lemma 5.1). Lemma 5.1 is the load-bearing step, so it was also tested numerically outside the formal development: for $k = 1, 2, 3$ and $N = 5, 7$, an exact rational fit of S_k in the spanning set $\{\beta^a P_{a_1} \cdots P_{a_l} : a + \sum a_i \leq 2k\}$ (of sizes 7, 26, 75) was computed from $|\text{basis}| + 5$ random samples (integer nodes in $[-25, 25)$, half-integer β) and verified on 10 fresh samples in each of the six cases; a violation of the degree bound would have appeared as an inconsistent linear system, and none did.

6. The third power sum

The source gives no formula for Σ_3; Question 3.7 asks only whether one of weight at most 8 exists. Raising the truncation of Section 3 from v^5 to v^7 gives $\widetilde{\phi}^{[d-6]}$ and hence Σ_3 by (17). The formula below was found as follows, in exact rational arithmetic. For each partition, Σ_3 was evaluated at $\beta = \frac{1}{2}, \ldots, \frac{13}{2}$ and interpolated exactly (its β-degree is at most 6 by Theorem 2.2), giving functions $c_0(\lambda), \ldots, c_6(\lambda)$; each c_a was fitted exactly over $\mathbb{Q}$ in the monomial basis of $\Lambda^*_{\leq 8-a}$ — the space Question 3.7 predicts, of total dimension $\sum_{a=0}^{8} p(8-a) = 67$ — from the 138 partitions of size 1 to 10; and the resulting

expression was verified on all 369 partitions with $11 \le |\lambda| \le 14$ before being written down. It fits in every one of the seven degrees:

$$\begin{aligned}\sum_{i=1}^{|\lambda|}\big(z_i^{(\beta)}(\lambda)\big)^3 = \Big(&132\mathsf{p}_8 + 651\mathsf{p}_6 - 200\mathsf{p}_4^2 - 504\mathsf{p}_2\mathsf{p}_6 + 960\mathsf{p}_2^2\mathsf{p}_4 - 352\mathsf{p}_2^4 + 800\mathsf{p}_2^3 - 1640\mathsf{p}_2\mathsf{p}_4 \\ &+ \tfrac{2303}{4}\mathsf{p}_4 - 492\mathsf{p}_2^2 + \tfrac{1125}{16}\mathsf{p}_2\Big) \\ &+ \beta\Big(462\mathsf{p}_7 - 1260\mathsf{p}_2\mathsf{p}_5 - 600\mathsf{p}_3\mathsf{p}_4 + 1440\mathsf{p}_2^2\mathsf{p}_3 + \tfrac{3255}{2}\mathsf{p}_5 - 2460\mathsf{p}_2\mathsf{p}_3 + \tfrac{6909}{8}\mathsf{p}_3\Big) \\ &+ \beta^2\Big(630\mathsf{p}_6 - 1320\mathsf{p}_2\mathsf{p}_4 - 450\mathsf{p}_3^2 + 544\mathsf{p}_2^3 + 1435\mathsf{p}_4 - 960\mathsf{p}_2^2 + \tfrac{2807}{8}\mathsf{p}_2\Big) \\ &+ \beta^3\big(420\mathsf{p}_5 - 720\mathsf{p}_2\mathsf{p}_3 + 525\mathsf{p}_3\big) + \beta^4\big(140\mathsf{p}_4 - 108\mathsf{p}_2^2 + 70\mathsf{p}_2\big) + 21\beta^5\mathsf{p}_3 + \beta^6\mathsf{p}_2.\end{aligned} \tag{22}$$

Theorem 6.1 (the third power sum, $|\lambda| \le 14$)**.** *For every partition λ with $1 \le |\lambda| \le 14$ and every $\beta \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{13}{2}\}$, the Vandermonde control, the cleared form of* (6) *and the cleared form of* (22) *(the third identity of* (20)*, computed in $\mathbb{Z}[v]/(v^7)$) hold. Consequently, by the β-degree bound of* [1, Theorem 2.2]*, identity* (22) *holds at every such λ as an identity of polynomials in β.*

Proof. Exhaustive evaluation inside the formal development, 7 determinants per partition over the 507 partitions of size 1 to 14. Both sides of (22) have β-degree at most 6 ($\widetilde{\phi}^{[d-2j]}$ has degree at most $2j$), so seven points decide. Exact rational arithmetic outside the development confirms (22) for every partition with $|\lambda| \le 16$ as well. □

Three things follow. Identity (22) answers the $k = 3$ case of Question 3.7 affirmatively at the predicted weight $2k + 2 = 8$, for $|\lambda| \le 14$ (≤ 16 with the exact-arithmetic range), and every one of its seven β-degrees uses only monomials of the predicted weight. It is not yet a theorem for all λ: Theorem 1.2 gives $\Sigma_3 \in \Lambda^*[\beta]_{\le 24}$, so the argument of Section 5 would need (22) for every partition of size at most 23 (5,763 partitions including the empty one, seven values of β each, about 38 minutes of exact arithmetic at the measured 56 ms per determinant) together with the shifted Schur vanishing theorem (21) for the injectivity, a rank certificate for $\Lambda^*_{\le 24}$ being a 1575×5763 elimination. And its top-degree layer is the one predicted before the fit:

Observation 6.2 (the top-degree layer; not machine-checked as a pattern)**.** Grade by $\deg\beta = 1$, $\deg \mathsf{p}_k = k$ and read off the coefficients of $(\mathsf{p}_{2k+2}, \beta\mathsf{p}_{2k+1}, \dots, \beta^{2k}\mathsf{p}_2)$ in Σ_k:

k	statement	top-degree coefficients
0	$\Sigma_0 = \mathsf{p}_2$	1
1	(5), Proposition 3.6 of [1]	2, 3, 1
2	(6), Theorem 1.1	14, 35, 30, 10, 1
3	(22), Theorem 6.1	132, 462, 630, 420, 140, 21, 1

Reversed, these are rows $0, 2, 4, 6$ of the triangle OEIS A088617 [8], $T(n, j) = \binom{n+j}{n}\binom{n}{j}/(j+1)$ $(0 \le j \le n)$, whose entry records that it counts Schröder paths by their number of up steps and that its row sums are the large Schröder numbers A006318: rows $[1]$, $[1, 3, 2]$, $[1, 10, 30, 35, 14]$, $[1, 21, 140, 420, 630, 462, 132]$. The leading coefficient of p_{2k+2} is $T(2k, 2k) = C_{2k}$, the Catalan number $(1, 2, 14, 132)$, and the coefficient of $\beta^{2k}\mathsf{p}_2$ is $T(2k, 0) = 1$.

The pattern is not in [1], which contains no occurrence of "Catalan", "Schröder" or "A088617", and we found it nowhere else. Row 6 was written down from the triangle before the fit of (22) was run — its two cheapest predictions, $\beta^6\mathsf{p}_2$ and $21\beta^5\mathsf{p}_3$, were first confirmed by hand at $\lambda = (2,1), (3,1), (2,2,1), (4,2,1)$ — and all seven coefficients came out as predicted. It is a pattern in four data points, and it is the natural next falsification target: if it continues, the top layer of Σ_4 is row 8 reversed, $1430\mathsf{p}_{10} + 6435\beta\mathsf{p}_9 + 12012\beta^2\mathsf{p}_8 + 12012\beta^3\mathsf{p}_7 + 6930\beta^4\mathsf{p}_6 + 2310\beta^5\mathsf{p}_5 + 420\beta^6\mathsf{p}_4 + 36\beta^7\mathsf{p}_3 + \beta^8\mathsf{p}_2$, with $C_8 = 1430$ in front, to be tested by a fit inside $\Lambda^*[\beta]_{\le 10}$ (dimension 139) from the v^9 truncation.

Remark 6.3 (what remains). The gap between the proved $\Lambda^*[\beta]_{\le 8k}$ and the conjectured $\Lambda^*[\beta]_{\le 2k+2}$ is the remaining content of Question 3.7; $k = 2$ is settled ($= 6$) and $k = 3$ is settled numerically ($= 8$). A

bound $\deg S_k \le k$ in Lemma 5.1 in place of $2k$, if true, would give weight $\le 4k$ by the same argument, and the observed layer structure suggests that the bound $2k+2$ is tight: the top layer is completely filled, with C_{2k} in front of p_{2k+2}. Closed forms for $\widetilde{\phi}^{[d-3]}$ and $\widetilde{\phi}^{[d-4]}$ as symmetric polynomials in the nodes (Lemma 5.1 says they exist, of degree ≤ 6 and ≤ 8; fits were verified to exist at $N = 5, 7$ but their N-dependence was not determined) would extend Theorem 2.2 of [1] by two orders and give a second, formula-level proof of Theorem 1.1.

7. Verification

Propositions 4.1, 4.2, 4.6 and Theorems 4.3, 4.4, 4.5, 5.4, 6.1 were verified in Lean 4 (toolchain v4.32.0 [9]); their statements are reproduced verbatim in Appendix A. The development has ten modules. Nine import only the core library and a one-attribute tagging module of the surrounding repository: `Core` (the truncated ring $\mathbb{Z}[v]/(v^5)$, the Bareiss determinant, the partition generator, the scaled shifted symmetric functions `bigP`, the nodes, the matrix, `okAt`, `checkLam`, `checkSize`, `checkSize5`), `Verify` ($|\lambda| \le 13$), `VerifyB` (14, 15), `VerifyC` (16), `VerifyD` (17), `Verify5` (18 to 21 at five points), `Cube` (the v^7 truncation and Theorem 6.1), `Reduction` (Theorem 5.4) and `Controls` (Proposition 4.6); `Equivalence` (Theorem 4.5) is the one module importing Mathlib [10]. Counting the sources, they hold 37 theorem declarations (13, 2, 1, 1, 4, 6, 2, 6, 2 in `Verify`, `VerifyB`, `VerifyC`, `VerifyD`, `Verify5`, `Cube`, `Reduction`, `Controls`, `Equivalence`) and 62 definitions (two of them structure declarations), all of which serve the statements of this paper. The development holds no data: partitions are generated, Laguerre coefficients computed from (7), matrices built and determinants taken inside Lean; the only numbers typed in are the coefficients of (5), (6), (12), (22) and of the control statements.

The two identities of Theorem 4.5 are proved by `ring` and depend only on `propext`, `Classical.choice` and `Quot.sound`. Every other theorem is an exhaustive evaluation discharged by `native_decide`, so it carries its own axiom `<decl>._native.native_decide.ax_1_1`, asserting that the compiled evaluation returned `true`: the sweeps `check_1`–`check_17`, `check5_18`–`check5_21`, `cube_le_9`–`cube_14` and the controls `pair_only_conjugate`, `mutilations_fail` together with `propext` and `Quot.sound`; `monomials_independent` and `rank_508` also with `Classical.choice`; `dims_16`, `parts_count`, `points_nonvacuous` with the native axiom alone; and `data_nontrivial` with `propext`. There is no `sorry` and no axiom declared by hand. One pass over the ten modules, one module at a time on one core, took `Core` 8 s, `Verify` 142 s, `VerifyB` 381 s, `VerifyC` 329 s, `VerifyD` 607 s, `Verify5` 315 s, `Cube` 61 s, `Reduction` 16 s, `Controls` 15 s and `Equivalence` 8 s of elapsed time, with peak resident memory at most 1.61 GB for the Mathlib-free modules and 6.98 GB for `Equivalence` (the loaded Mathlib image), so about 1,880 s in all. The exact-arithmetic Python runs of Table 1, Theorem 4.4's extension to $|\lambda| \le 25$, the fit and verification of (22) and the controls of Remark 3.1 take about 5,140 s between them, so the computations reported in this paper cost a little under two processor-hours; the whole founding computation, including the exploratory runs and the literature searches that are not itemised here, is recorded at about 2.6 processor-hours. Theorems 1.1 and 1.2 and Lemmas 5.1–5.3 are proved on paper only; their finite ingredients are the machine-checked statements above. The source of the development and of the Python scripts is currently held in a private repository; repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted), preceded by the definitions that carry the content of the statements, grouped by module; the remaining auxiliary definitions are described here in words instead. Partitions are weakly decreasing lists of naturals; `partsOf n` lists those of n (first (1^n), last (n)) and `conjOf` conjugates; `ipow` is integer exponentiation; `bigP lam k` is $P_k = 2^{k-1}\mathsf{p}_k(\lambda)$; a parameter `T` is 2β; `bareissDet N M` is the fraction-free determinant of the row-major $N \times N$ array `M` over $\mathbb{Z}[v]/(v^5)$ (`Ser`, five integer coefficients), `blzMatrix N X t c` the matrix (14) with monomial rows $(X_i - 2p - c)^k$ and entries scaled by 6144, and `vandermonde N X` is $\prod_{i<j}(X_j - X_i)$; `bareissDet7` and `blzMatrix7` are the same over $\mathbb{Z}[v]/(v^7)$ (`Ser7`) with scale 2949120. In `Reduction`, `invMod` is the inverse modulo `pmod` and `rowCombine a b f` is $a - fb$ modulo `pmod`.

Core.lean

```
def bigP (lam : List Nat) (k : Nat) : Int := Id.run do
  let mut acc : Int := 0
  let mut i : Nat := 1
  for li in lam do
    acc := acc + ipow (2 * (li : Int) - 2 * (i : Int) + 1) (k - 1)
               - ipow (1 - 2 * (i : Int)) (k - 1)
    i := i + 1
  return acc

def rhsNum (P2 P3 P4 P5 P6 T : Int) : Int :=
  2 * (21 * P6 - 120 * P4 * P2 + 128 * P2 * P2 * P2 + 130 * P4 - 240 * P2 * P2 + 81 * P2)
  + 3 * T * (35 * P5 - 120 * P3 * P2 + 130 * P3)
  + 6 * T * T * (15 * P4 - 22 * P2 * P2 + 25 * P2)
  + 30 * T * T * T * P3
  + 3 * T * T * T * T * P2

def propNum (P2 P3 P4 T : Int) : Int :=
  2 * P4 - 4 * P2 * P2 + 3 * T * P3 + T * T * P2 + 2 * P2

def scP : Nat → Int
  | 0 => 6144
  | 1 => -1536
  | 2 => 192
  | 3 => -16
  | _ => 1

def gCoef (t X : Int) (p : Nat) : Int := Id.run do
  let mut acc : Int := scP p
  for l in [0:p] do
    acc := acc * (X - 2 * (l : Int)) * (X - t - 2 * (l : Int))
  return acc

def nodesOf (lam : List Nat) (T : Int) : Array Int × Int := Id.run do
  let r := lam.length
  let s := lam.headD 0
  let lc := conjOf lam
  let t : Int := T - 2 * (r : Int) + 2 * (s : Int)
  let mut X : Array Int := Array.emptyWithCapacity (r + s)
  for i in [0:r] do
    X := X.push (2 * ((lam.getD i 0 : Int) + (r : Int) - 1 - (i : Int)) + t)
  for j in [0:s] do
    X := X.push (2 * ((lc.getD j 0 : Int) + (s : Int) - 1 - (j : Int)))
  return (X, t)

def phis (lam : List Nat) (T : Int) : Int × Int × Int × Int :=
  let (X, t) := nodesOf lam T
  let N := X.size
  let c := Id.run do
    let mut lo := X[0]!
    let mut hi := X[0]!
    for i in [0:N] do
      if X[i]! < lo then lo := X[i]!
      if X[i]! > hi then hi := X[i]!
    return (lo + hi) / 2
  let D := bareissDet N (blzMatrix N X t c)
  (D.c0, D.c2, D.c4, ipow 6144 N * vandermonde N X)

def okAt (P2 P3 P4 P5 P6 : Int) (lam : List Nat) (T : Int) : Bool :=
  let (p0, p2, p4, kap) := phis lam T
  p0 != 0 && p0 == kap
    && (-8) * p2 == propNum P2 P3 P4 T * p0
    && 96 * (p2 * p2 - 2 * p4 * p0) == rhsNum P2 P3 P4 P5 P6 T * (p0 * p0)

def checkLam (lam : List Nat) : Bool :=
  let r := lam.length
  let s := lam.headD 0
  let P2 := bigP lam 2
  let P3 := bigP lam 3
  let P4 := bigP lam 4
  let P5 := bigP lam 5
  let P6 := bigP lam 6
  (List.range (2 * r * s + 5)).all (fun j => okAt P2 P3 P4 P5 P6 lam (2 * (j : Int) + 1))

def checkLam5 (lam : List Nat) : Bool :=
```

```
  let P2 := bigP lam 2
  let P3 := bigP lam 3
  let P4 := bigP lam 4
  let P5 := bigP lam 5
  let P6 := bigP lam 6
  ([1, 3, 5, 7, 9] : List Int).all (fun T => okAt P2 P3 P4 P5 P6 lam T)

def checkSize (n : Nat) : Bool := (partsOf n).all checkLam

def checkSize5 (n : Nat) : Bool := (partsOf n).all checkLam5
```

`Verify.lean`, `VerifyB.lean`, `VerifyC.lean`, `VerifyD.lean` (Propositions 4.1, 4.2, Theorem 4.3; `check_1` to `check_13` in `Verify`, `check_14`, `check_15` in `VerifyB`, `check_16` in `VerifyC`, `check_17` in `VerifyD`)

```
theorem check_1 : checkSize 1 = true
theorem check_2 : checkSize 2 = true
theorem check_3 : checkSize 3 = true
theorem check_4 : checkSize 4 = true
theorem check_5 : checkSize 5 = true
theorem check_6 : checkSize 6 = true
theorem check_7 : checkSize 7 = true
theorem check_8 : checkSize 8 = true
theorem check_9 : checkSize 9 = true
theorem check_10 : checkSize 10 = true
theorem check_11 : checkSize 11 = true
theorem check_12 : checkSize 12 = true
theorem check_13 : checkSize 13 = true
theorem check_14 : checkSize 14 = true
theorem check_15 : checkSize 15 = true
theorem check_16 : checkSize 16 = true
theorem check_17 : checkSize 17 = true
```

`Verify5.lean` (Theorem 4.4)

```
theorem check5_18 : checkSize5 18 = true
theorem check5_19 : checkSize5 19 = true
theorem check5_20 : checkSize5 20 = true
theorem check5_21 : checkSize5 21 = true
```

`Cube.lean` (Theorem 6.1)

```
def scP7 : Nat → Int
  | 0 => 2949120
  | 1 => -737280
  | 2 => 92160
  | 3 => -7680
  | 4 => 480
  | 5 => -24
  | _ => 1

def gCoef7 (t X : Int) (p : Nat) : Int := Id.run do
  let mut acc : Int := scP7 p
  for l in [0:p] do
    acc := acc * (X - 2*(l : Int)) * (X - t - 2*(l : Int))
  return acc

def phis6 (lam : List Nat) (T : Int) : Int × Int × Int × Int × Int :=
  let (X, t) := nodesOf lam T
  let N := X.size
  let c := Id.run do
    let mut lo := X[0]!
    let mut hi := X[0]!
    for i in [0:N] do
      if X[i]! < lo then lo := X[i]!
      if X[i]! > hi then hi := X[i]!
    return (lo + hi) / 2
  let D := bareissDet7 N (blzMatrix7 N X t c)
  (D.c0, D.c2, D.c4, D.c6, ipow 2949120 N * vandermonde N X)

def cubeNum (P2 P3 P4 P5 P6 P7 P8 T : Int) : Int :=
  (-2816)*P2*P2*P2*P2 + 3840*P2*P2*P4 - 1008*P2*P6 - 400*P4*P4 + 132*P8
  + 12800*P2*P2*P2 - 13120*P2*P4 + 2604*P6 - 15744*P2*P2 + 9212*P4 + 4500*P2
  + 5760*T*P2*P2*P3 - 2520*T*P2*P5 - 1200*T*P3*P4 + 462*T*P7 - 19680*T*P2*P3
  + 6510*T*P5 + 13818*T*P3
  + 2176*T*T*P2*P2*P2 - 2640*T*T*P2*P4 - 900*T*T*P3*P3 + 630*T*T*P6
  - 7680*T*T*P2*P2 + 5740*T*T*P4 + 5614*T*T*P2
  - 1440*T*T*T*P2*P3 + 420*T*T*T*P5 + 2100*T*T*T*P3
  - 216*T*T*T*T*P2*P2 + 140*T*T*T*T*P4 + 280*T*T*T*T*P2
```

```
  + 21*T*T*T*T*T*P3
  + T*T*T*T*T*T*P2

def cubeOkAt (P2 P3 P4 P5 P6 P7 P8 : Int) (lam : List Nat) (T : Int) : Bool :=
  let (p0, p2, p4, p6, kap) := phis6 lam T
  p0 != 0 && p0 == kap
    && 96 * (p2*p2 - 2*p4*p0) == rhsNum P2 P3 P4 P5 P6 T * (p0*p0)
    && 128 * (-(p2*p2*p2) + 3*p2*p4*p0 - 3*p6*p0*p0) == cubeNum P2 P3 P4 P5 P6 P7 P8 T * (p0*p0*p0)

def cubeCheckLam (lam : List Nat) : Bool :=
  let P2 := bigP lam 2
  let P3 := bigP lam 3
  let P4 := bigP lam 4
  let P5 := bigP lam 5
  let P6 := bigP lam 6
  let P7 := bigP lam 7
  let P8 := bigP lam 8
  (List.range 7).all (fun j => cubeOkAt P2 P3 P4 P5 P6 P7 P8 lam (2*(j : Int) + 1))

def cubeCheckSize (n : Nat) : Bool := (partsOf n).all cubeCheckLam

theorem cube_le_9 : (List.range 9).all (fun n => cubeCheckSize (n+1)) = true
theorem cube_10 : cubeCheckSize 10 = true
theorem cube_11 : cubeCheckSize 11 = true
theorem cube_12 : cubeCheckSize 12 = true
theorem cube_13 : cubeCheckSize 13 = true
theorem cube_14 : cubeCheckSize 14 = true
```

`Reduction.lean` (Theorem 5.4)

```
def pmod : Int := 2147483647

def allParts (m : Nat) : List (List Nat) := (List.range (m + 1)).flatMap partsOf

def monoList (m : Nat) : List (List Nat) := (allParts m).filter (fun mu => mu.all (fun x => decide (2 ≤ x)))

def evalMatrix (m ncols : Nat) : Array (Array Int) := Id.run do
  let lams := ((allParts m).take ncols).toArray
  -- per column, the values `bigP λ j` for `j ≤ m`
  let mut cols : Array (Array Int) := Array.emptyWithCapacity lams.size
  for lam in lams do
    let mut v : Array Int := Array.emptyWithCapacity (m + 1)
    for j in [0:m+1] do
      v := v.push (bigP lam j % pmod)
    cols := cols.push v
  let mut rows : Array (Array Int) := Array.emptyWithCapacity (monoList m).length
  for mu in monoList m do
    let mut row : Array Int := Array.emptyWithCapacity lams.size
    for ci in [0:lams.size] do
      let v := cols[ci]!
      let mut acc : Int := 1
      for j in mu do
        acc := acc * v[j]! % pmod
      row := row.push acc
    rows := rows.push row
  return rows

def rankMod (nrows ncols : Nat) (R0 : Array (Array Int)) : Nat := Id.run do
  let mut R := R0
  let mut r : Nat := 0
  for c in [0:ncols] do
    if r < nrows then
      let mut piv := nrows
      for i in [r:nrows] do
        if piv == nrows then
          if R[i]![c]! % pmod != 0 then piv := i
      if piv != nrows then
        if piv != r then
          let a := R[r]!
          let b := R[piv]!
          R := R.set! r b
          R := R.set! piv a
        let inv := invMod (R[r]![c]!)
        R := R.set! r ((R[r]!).map (fun x => x * inv % pmod))
        let rowr := R[r]!
        for i in [r+1:nrows] do
```

```
          let f := R[i]![c]! % pmod
          if f != 0 then
            R := R.set! i (rowCombine (R[i]!) rowr f)
        r := r + 1
  return r

theorem dims_16 : (monoList 16).length = 231 ∧ (allParts 16).length = 915
theorem monomials_independent : rankMod 231 509 (evalMatrix 16 509) = 231
```

`Controls.lean` (Proposition 4.6, items (1)–(6) in order)

```
theorem parts_count :
    (List.range 13).map (fun n => (partsOf n).length)
      = [1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77]

theorem points_nonvacuous :
    (partsOf 6).all (fun lam => decide (7 ≤ 2 * lam.length * lam.headD 0 + 5)) = true

theorem data_nontrivial :
    (bigP [3,2,1] 2, bigP [3,2,1] 3, bigP [3,2,1] 4, bigP [3,2,1] 5, bigP [3,2,1] 6)
      = (12, 0, 252, 0, 6252)

def rhsNumNoB4 (P2 P3 P4 P5 P6 T : Int) : Int :=
  rhsNum P2 P3 P4 P5 P6 T - 3 * T * T * T * T * P2

def rhsNumNoP2 (P2 P3 P4 P5 P6 T : Int) : Int :=
  rhsNum P2 P3 P4 P5 P6 T - 2 * 81 * P2

def rhsNumBad14 (P2 P3 P4 P5 P6 T : Int) : Int :=
  rhsNum P2 P3 P4 P5 P6 T + 3 * P6

def okAtWith (f : Int → Int → Int → Int → Int → Int → Int)
    (P2 P3 P4 P5 P6 : Int) (lam : List Nat) (T : Int) : Bool :=
  let (p0, p2, p4, _) := phis lam T
  96 * (p2 * p2 - 2 * p4 * p0) == f P2 P3 P4 P5 P6 T * (p0 * p0)

def checkSizeWith (f : Int → Int → Int → Int → Int → Int → Int) (n : Nat) : Bool :=
  (partsOf n).all (fun lam =>
    let P2 := bigP lam 2
    let P3 := bigP lam 3
    let P4 := bigP lam 4
    let P5 := bigP lam 5
    let P6 := bigP lam 6
    (List.range 5).all (fun j => okAtWith f P2 P3 P4 P5 P6 lam (2 * (j : Int) + 1)))

theorem mutilations_fail :
    (checkSizeWith rhsNum 4, checkSizeWith rhsNumNoB4 4,
     checkSizeWith rhsNumNoP2 4, checkSizeWith rhsNumBad14 4)
      = (true, false, false, false)

def nodesOfPair (lam mu : List Nat) (T : Int) : Array Int := Id.run do
  let r := lam.length
  let s := mu.length
  let t : Int := T - 2 * (r : Int) + 2 * (s : Int)
  let mut X : Array Int := Array.emptyWithCapacity (r + s)
  for i in [0:r] do
    X := X.push (2 * ((lam.getD i 0 : Int) + (r : Int) - 1 - (i : Int)) + t)
  for j in [0:s] do
    X := X.push (2 * ((mu.getD j 0 : Int) + (s : Int) - 1 - (j : Int)))
  return X

def pairOk (lam mu : List Nat) (T : Int) : Bool :=
  let X := nodesOfPair lam mu T
  let N := X.size
  let t : Int := T - 2 * (lam.length : Int) + 2 * (mu.length : Int)
  let D := bareissDet N (blzMatrix N X t 0)
  let p0 := D.c0
  let p2 := D.c2
  let p4 := D.c4
  p0 != 0 &&
    96 * (p2 * p2 - 2 * p4 * p0)
      == rhsNum (bigP lam 2) (bigP lam 3) (bigP lam 4) (bigP lam 5) (bigP lam 6) T * (p0 * p0)

theorem pair_only_conjugate :
    ([[2,1,1], [1,1,1,1], [2,2], [3,1], [4]] : List (List Nat)).map
        (fun mu => ([1,3,5,7,9] : List Int).all (fun T => pairOk [3,1] mu T))
```

```
      = [true, false, false, false, false]

theorem rank_508 : rankMod 231 508 (evalMatrix 16 508) = 230
```

`Equivalence.lean` (Theorem 4.5)

```
def rhs35 (b p2 p3 p4 : ℚ) : ℚ := 2*p4 - 2*p2^2 + 3*b*p3 + b^2*p2 + p2/2

def rhs38 (b p2 p3 p4 p5 p6 : ℚ) : ℚ :=
  (14*p6 - 40*p4*p2 + (64/3)*p2^3 + (65/3)*p4 - 20*p2^2 + (27/8)*p2)
  + b*(35*p5 - 60*p3*p2 + (65/2)*p3)
  + b^2*(30*p4 - 22*p2^2 + (25/2)*p2)
  + 10*b^3*p3 + b^4*p2

def Iham1 (b p2 : ℚ) : ℚ := (b^2 - 1)/8 + p2

def Iham3 (b p2 p3 p4 : ℚ) : ℚ := ((b^2 - 1)/8)^2 + (3/4)*b^2*p2 + (3/2)*b*p3 + p4

def Iham5 (b p2 p3 p4 p5 p6 : ℚ) : ℚ :=
  ((b^2 - 1)/8)^3 + (15/64)*b^4*p2 + (15/16)*b^3*p3
  + b^2*((15/8)*p4 + (3/16)*p2) + b*((15/8)*p5 + (3/8)*p3) + (3/4)*p6 + p4/4

theorem rhs35_eq_cor77 (b p2 p3 p4 : ℚ) :
    rhs35 b p2 p3 p4 = 2 * Iham3 b p2 p3 p4 - 2 * (Iham1 b p2)^2

theorem rhs38_eq_rem78 (b p2 p3 p4 p5 p6 : ℚ) :
    rhs38 b p2 p3 p4 p5 p6
      = (56/3) * Iham5 b p2 p3 p4 p5 p6 - 40 * Iham3 b p2 p3 p4 * Iham1 b p2
        + (64/3) * (Iham1 b p2)^3 + 12 * Iham3 b p2 p3 p4 - 12 * (Iham1 b p2)^2
```

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