MAXIMAL-DETERMINANT RECORDS AT ORDERS 51, 107 AND 115, AND AN ADJUGATE OBSTRUCTION AT ORDER 7

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ABSTRACT. Let $D(n)$ be the largest $|\det X|$ over $X \in \{-1,+1\}^{n\times n}$ and $a(n)$ the largest $|\det M|$ over $M \in \{0,1\}^{n\times n}$ (OEIS A003432). Butbaia et al. recently announced new record lower bounds for $D(n)$ at $n = 51, 107, 115$, publishing for each order two ±1 seed strings, a block layout in two circulants, and the value $|\det X|/2^{\,n-1}$. We turn those three announcements into theorems. The matrices are rebuilt from the published seeds and nothing else; the determinants are then obtained from a triangular certificate — a lower triangular L with $L \cdot (X$ row-permuted) upper triangular — which replaces the $n!$-term Leibniz sum by about $n^3/2$ integer multiply–adds, so that a 115×115 determinant with a 119-digit value is confirmed in 29 seconds. A classical bordering identity, proved here for every n and every matrix, carries the three values to $a(50), a(106), a(114)$, three orders that A003432's b-file does not reach. We also record the complete multiset of entries of $XX^{\mathsf T}$ for each record matrix, data the announcement does not print, and observe that exactly one of the three constructions is normal. In a second part we prove that a matrix with constant row sum k satisfies $k\,g = n \det M$, where g is the grand sum of $\operatorname{adj} M$; the coprimality hypothesis of a conjecture of Baghel, Ravsky and Segal-Halevi then forces $\det M \mid k$, which shows that conjecture can never certify a maximal determinant at an order $4t-1$ carrying a Hadamard matrix of order $4t$. At $n = 7$, where a determinant of ±32 classically forces row sum 4, the grand sum is then always ±56 and never coprime to 32. Every numbered statement below is formally verified in Lean 4.

1. INTRODUCTION

The objects. For $n \geq 1$ put

$$D(n) \;=\; \max\bigl\{\,|\det X| \;:\; X \in \{-1,+1\}^{n\times n}\,\bigr\}, \qquad a(n) \;=\; \max\bigl\{\,|\det M| \;:\; M \in \{0,1\}^{n\times n}\,\bigr\}.$$

The two are the same problem in two normalizations: bordering a 0/1 matrix of order n with a row and a column of ones and replacing the body M by $J - 2M$ gives a ±1 matrix of order $n+1$, and this correspondence yields $D(n+1) = 2^n a(n)$. The sequence $a(n)$ is OEIS A003432 and $D(n)$ is A003433. Hadamard's inequality gives $D(n) \leq n^{n/2}$, equivalently $a(n) \leq 2^{-n}(n+1)^{(n+1)/2}$, with equality exactly when a Hadamard matrix of order $n+1$ exists, hence only when $n+1 \in \{1,2\}$ or $4 \mid n+1$ [4, 5]. Away from those orders the exact value is known only for small n: A003432 carries 22 terms, $a(0), \ldots, a(21)$, and its b-file stops there; correspondingly the maximum $D(n)$ is certified only up to $n = 22$ [8, 1]. For every larger n, and in particular for every order considered in Section 4, the true maximum is unknown and only lower bounds by explicit matrices are available.

The residue class $n \equiv 3 \pmod 4$ is the one on which, by the announcement's own account, the least progress has been made [1]. The orders in that class below 23 — namely $3, 7, 11, 15, 19$ — are settled, and beyond $n = 22$ the tabulated values are lower bounds attained by explicit constructions, none of which has been proved optimal.

The two sources. The first source is a recent announcement of three new records. Butbaia, Vipulanandan, Tan, Huang, Saunders-A'Court, Fagan, Passaro, Tarquini and Gukov [1] report improved lower bounds for $D(n)$ at $n = 51, 107, 115$, all three $\equiv 3 \pmod 4$. Their paper prints, per order, two ±1 seed strings of length $m = (n-1)/2$, a block layout built from the circulants they generate, and the normalized determinant $|\det X|/2^{\,n-1}$. Verification there is a computer-algebra script. The previous records at these three orders are quoted here from the maximal-determinant table maintained by Orrick, whose summary page was last modified on 17 October 2012 and has been offline since about 2020, so that it is reachable only through the Internet Archive [7]; the survey [3] records exactly this

disappearance and draws its own record table from the same source. That table covers $23 \le n \le 99$ in this residue class, and reports each order as a ratio to an upper bound rather than as an integer: its $n = 51$ entry, 0.6481, is the previous $n = 51$ record divided by the Barba bound $\sqrt{101} \cdot 50^{25}$, which we recompute as 0.648105 in Remark 4.4. Orders 107 and 115 lie outside it, and we have not found the integers O_{107}, O_{115} of Theorems 4.2–4.3 printed anywhere but on the archived pages and in [1].

The second source is a paper on fair electricity division. Baghel, Ravsky and Segal-Halevi [2] study k-times bin packing, where every item must be packed into k distinct bins, and attach to an $n \times n$ binary matrix A the pair $(\det A, g)$ with g the sum of all entries of $\det(A)A^{-1}$. Writing $K(n)$ for the largest k needed at n households, they prove $K(n) \le a(n)$ and $K(6) = 9$, and conjecture:

Conjecture 1.1 ([2], verbatim)**.** Let A be an $n \times n$ binary matrix, let $B := \det(A) \cdot A^{-1}$, and let g be the sum of all elements in B (the "grand sum" of B). If g and $\det(A)$ are coprime, then $K(n) \ge \det(A)$.

The paper adds that it is open whether its $K(6) = 9$ proposition generalizes. Since the conjecture's conclusion is strongest when $\det A$ is largest, the natural attempt is to feed it a matrix attaining $a(n)$; the order-7 case, where $a(7) = 32$, is the first one where that attempt is not immediately settled by hand.

What is settled here. Section 3 isolates the certificate that makes the record layer affordable: four checkable hypotheses on a lower triangular matrix L pin $|\det X|$ without any appeal to the Leibniz sum. The argument is Bareiss's fraction-free elimination read backwards and is classical; what is new is the use, since it turns a determinant of order 115 — a sum over 115! permutations — into $n^3/2$ integer multiply–adds inside a proof assistant.

Section 4 states the three records as theorems: for each order the matrix is rebuilt from the two published seed strings by the published circulant convention and block layout, and its determinant is exactly the published value. Nothing is copied but the six strings, whose lengths are themselves checked. Section 5 proves the bordering identity for every n and transfers the three values to record lower bounds for $a(50), a(106), a(114)$, none of which occurs in A003432. Section 6 records the Gram profiles.

Sections 7–8 treat the second source. The single identity $k\,g = n \det M$ for a matrix of constant row sum k turns the coprimality hypothesis of Conjecture 1.1 into the divisibility $\det A \mid k$. At an order $n = 4t - 1$ carrying a Hadamard matrix of order $4t$ every matrix attaining $a(n)$ has row sum $k = 2t$, so the conjecture — even if true — can never certify more than $K(n) \ge 2t$ there, against a maximal determinant that is exponentially larger. At $n = 7$ this is 4 against $a(7) = 32$, and the accompanying question, whether some 7×7 binary matrix of determinant 32 has an odd grand sum, is answered in the negative: such a matrix has row sum 4 — classically, and this is the one input we do not formalize — and its grand sum is then always ± 56.

We prove nothing about Conjecture 1.1 itself, and we compute no value of $K(n)$. We prove no upper bound for $D(n)$ at $n = 51, 107, 115$; those remain open.

2. Notation

All matrices are square with integer entries and indices in $\{0, \dots, n-1\}$. We write $\operatorname{adj} M$ for the adjugate, so that $(\operatorname{adj} M)M = M(\operatorname{adj} M) = (\det M)I$ with no invertibility hypothesis, and

$$g(M) \;=\; \sum_i \sum_j (\operatorname{adj} M)_{ij}$$

for its *grand sum*. When M is invertible, $\operatorname{adj} M = \det(M)M^{-1}$, so $g(M)$ is exactly the quantity called g in Conjecture 1.1. We write $\mathbf{1}$ for an all-ones column vector, J for an all-ones matrix, and I for an identity matrix, of whatever size the context demands. A matrix has *row sum* k if $M\mathbf{1} = k\,\mathbf{1}$.

For a string $c = (c_0, \dots, c_{m-1})$ of signs, $\operatorname{circ}(c)$ is the $m \times m$ circulant with

$$\operatorname{circ}(c)_{ij} \;=\; c_{(j-i) \bmod m},$$

which is the convention of [1]. For a permutation σ of the row indices, X_σ denotes the matrix with $(X_\sigma)_{ij} = X_{\sigma(i)j}$, so that $\det X_\sigma = \operatorname{sgn}(\sigma) \det X$.

3. A triangular certificate for a large determinant

The determinant of a matrix of order 115 cannot be obtained from the definition: the Leibniz sum has 115! terms and cofactor expansion is worse. What follows replaces the definition by four hypotheses that a machine can check in $O(n^3)$ integer operations.

Theorem 3.1. *Let X, L be $n \times n$ integer matrices, σ a permutation of $\{0, \dots, n-1\}$ and D an integer. Put $Y = X_\sigma$. Assume*

(1) $L_{ij} = 0$ *whenever* $i < j$;
(2) $(LY)_{ij} = 0$ *whenever* $j < i$;
(3) $\prod_i L_{ii} \neq 0$;
(4) $\prod_i (LY)_{ii} = \left(\prod_i L_{ii}\right) \cdot D$.

Then $|\det X| = |D|$.

Proof. By (1) and (2) both L and LY are triangular, so $\det L = \prod_i L_{ii}$ and $\det(LY) = \prod_i (LY)_{ii}$. Multiplicativity of the determinant and $\det Y = \operatorname{sgn}(\sigma) \det X$ give

$$\left(\prod_i L_{ii}\right) \cdot D \;=\; \det(LY) \;=\; \det L \cdot \det Y \;=\; \left(\prod_i L_{ii}\right) \cdot \operatorname{sgn}(\sigma) \det X.$$

Cancelling the nonzero factor $\prod_i L_{ii}$ in the integral domain $\mathbb{Z}$ leaves $D = \operatorname{sgn}(\sigma) \det X$, and $|\operatorname{sgn}(\sigma)| = 1$. □

Nothing about the origin of L enters the statement, which is why the certificate is checked rather than trusted: an incorrect L cannot make the four hypotheses hold with an incorrect D. Section 4.3 exhibits, for each order, a right L refusing a wrong D, and shows that hypotheses (2) and (3) are both indispensable.

Remark 3.2 (where L comes from). Bareiss's fraction-free elimination [6] run on the augmented matrix $[\,Y \mid I\,]$ produces exactly such an L whenever the leading principal minors of Y are nonzero: the rows of L are the integer combinations of rows of Y that the elimination forms, L_{ii} is the $(i-1)$-st leading principal minor $d_{i-1}(Y)$ and $(LY)_{ii} = d_i(Y)$, so hypothesis (4) is the telescoping identity $\left(\prod_{i<n} d_i\right) \det Y = \prod_{1 \le i \le n} d_i$. Each entry of L is up to sign a minor of a ± 1 matrix and so at most $n^{n/2}$ in absolute value. Zero pivots do occur, and a row swap applied to the augmented identity destroys lower-triangularity of L; the remedy used here is two passes, one to discover the pivoting order and one to re-run on the already permuted matrix with no swaps at all. That permutation is σ, and since only $|\det X|$ is claimed its sign is never needed. All of this is provenance: the theorem assumes none of it.

Remark 3.3 (cost). A certificate for order n is $n(n+1)/2$ integers. Only the entries of LY on and below the diagonal are ever evaluated, and L is stored so that the entries above its diagonal are zero by definition rather than by computation; the check is therefore about $n^3/2$ integer multiply–adds on operands of at most $n^{n/2}$. For the three orders of Section 4 the certificates hold 1326, 5778 and 6670 integers (0.031, 0.350 and 0.452 megabytes of decimal digits), and the verified checks took 11.3, 18.3 and 29.0 seconds of wall time, at a peak memory of at most half a gigabyte above the cost of loading the ambient library. Producing all three certificates outside the proof assistant took 6.7 seconds.

4. Three record determinants

4.1. **The constructions.** Fix $n \in \{51, 107, 115\}$ and $m = (n-1)/2$, and let $a, b \in \{-1, +1\}^m$ be the two seed strings of [1] for that order, transcribed in Table 1 with $+$ for $+1$ and $-$ for -1. Put

$A = \mathrm{circ}(a)$ and $B = \mathrm{circ}(b)$. The matrices are

$$X_{51} = \begin{pmatrix} A & B & -\mathbf{1} \\ B & A^{\mathsf{T}} & \mathbf{1} \\ \mathbf{1}^{\mathsf{T}} & -\mathbf{1}^{\mathsf{T}} & 1 \end{pmatrix}, \quad X_{107} = \begin{pmatrix} 1 & \mathbf{1}^{\mathsf{T}} & -\mathbf{1}^{\mathsf{T}} \\ \mathbf{1} & A & B \\ -\mathbf{1} & B^{\mathsf{T}} & -A^{\mathsf{T}} \end{pmatrix}, \quad X_{115} = \begin{pmatrix} A & B & -\mathbf{1} \\ B^{\mathsf{T}} & -A^{\mathsf{T}} & \mathbf{1} \\ -\mathbf{1}^{\mathsf{T}} & -\mathbf{1}^{\mathsf{T}} & -1 \end{pmatrix}.$$

Each of these three displays is the corresponding display of [1, §§2.1–2.3, v1], read in the transpose convention of Section 2: the announcement writes its all-ones vector as a *row* j_n of length m, so its j_n is our $\mathbf{1}^{\mathsf{T}}$ and its j_n^{T} is our $\mathbf{1}$. We checked the layouts, and the six seed strings, against the LaTeX source of the e-print itself.

n	seeds a, b
51	`-+-++--+--++++------+---+`
	`----+---+--+-+--+-++--+++`
107	`-+-+-++++-+--++-+-++++--++++-++--------+-++-++------++`
	`++-+-+---++++-+-++++-++-++----+---++--++-++-+++-+---+`
115	`++++-++-++--+-++---+---++-+++-+++++-+-----++--+-++++--+-`
	`-+-+---++-+--+-+--+++++--+-+-+++++-+--++--+++---++-----+`

TABLE 1. The six seed strings of [1], the only data taken from that paper. Their lengths $25, 25, 53, 53, 57, 57$ are part of the verified statement (Section 4.3).

4.2. **The determinants.** Each of the following three theorems is a statement about the explicit matrix just defined.

Theorem 4.1 (order 51)**.** *X_{51} has all entries ± 1 and $|\det X_{51}| = N_{51} \cdot 2^{50}$, a 44-digit integer, where N_{51} is the 29-digit integer*

$$N_{51} = 17776121037665193653653203125.$$

In particular $D(51) \geq N_{51} \cdot 2^{50}$, which exceeds $O_{51} \cdot 2^{50}$ by about 3.11%, where

$$O_{51} = 17240748636787425652496176509$$

is the previously recorded value.

Theorem 4.2 (order 107)**.** *X_{107} has all entries ± 1 and $|\det X_{107}| = N_{107} \cdot 2^{106}$, a 109-digit integer, where N_{107} is the 77-digit integer*

25405109779472820154713362533412847329846084693257600588972842045966418068198.

In particular $D(107) \geq N_{107} \cdot 2^{106}$, which exceeds $O_{107} \cdot 2^{106}$ by about 0.44%, where O_{107} is the previously recorded value

252947317470812382204190324461395480361542526022262123455396358978073532225 02.

Theorem 4.3 (order 115)**.** *X_{115} has all entries ± 1 and $|\det X_{115}| = N_{115} \cdot 2^{114}$, a 119-digit integer, where N_{115} is the 84-digit integer*

824875559997507123862490321617482346789417543713732896000289208985971332478680569108.

In particular $D(115) \geq N_{115} \cdot 2^{114}$, which exceeds $O_{115} \cdot 2^{114}$ by about 1.68%, where O_{115} is the previously recorded value

81124133962505253495817162333640570825929027091466275074070690426107826294749520148 8.

Proof of Theorems 4.1–4.3. In each case the matrix X is the one displayed above, assembled from the two seed strings; the ± 1 claim is the evaluation of its n^2 entries. A lower triangular L and a row permutation σ were produced by two-pass Bareiss elimination as in Remark 3.2 and are carried with the proof; the four hypotheses of Theorem 3.1 are then verified by compiled evaluation — $n(n+1)/2$ entries of LY below the diagonal, n diagonal entries, and two products of n integers — with $D = -200141330203304502073615340922522828800000000$ at $n = 51$, and the analogous integers

at 107 (negative) and 115 (positive). Theorem 3.1 converts that into $|\det X| = |D|$, and $|D|$ is the stated multiple of 2^{n-1}. The permutation moves 15 positions at $n = 51$, none at $n = 107$ — the elimination meets no zero pivot there — and 4 at $n = 115$. The comparisons with the previous records are arithmetic between two explicit integers. □

Two remarks on what these statements do and do not contain. First, the numbers N_{51}, N_{107}, N_{115} are the ones printed in [1]; they are not obtained here by any search. What is established is that the matrices built from the published seeds really do have those determinants, hence that the announced lower bounds for $D(n)$ hold. Before the formal development the same three values were reproduced from the seeds by an independent fraction-free implementation in 0.52 seconds, with 2^{n-1} dividing each exactly and the improvement ratios matching the percentages quoted in [1]; the two computations use different code and different arithmetic, and they agree.

Second, that O_{51}, O_{107}, O_{115} are the *previous* records is a bibliographic claim, not a theorem. It rests on the archived per-order pages of Orrick's table, read on 29 August 2026, each of which prints the value, credits Tomas Rokicki, and adds "This form has not been proved to be optimal" [7]; the same three integers are quoted as the old records in [1]. Nothing here bounds $D(n)$ from above at these orders.

The lineage of the constructions is worth one sentence, since it locates what is and is not new about the matrices themselves. The announcement describes its parametrizations as similar to those of Brent and Yedidia [11] and of Koukouvinos, Kounias and Seberry [10]. Brent and Yedidia tabulate the maximal determinant of a *single* binary circulant — a different and much smaller quantity than $D(n)$ — for orders $n \le 53$ in the 0/1 normalization and $n \le 52$ in the ± 1 one. The previous systematic improvement of the record table, by Orrick, Solomon, Dowdeswell and Smith [9], names in its abstract the orders it improves $(22, 23, 27, 29, 31, 33, 34, 35, 39, 45, 47, 53, 63, 69, 73, 77, 79, 93, 95)$ and the orders it tabulates for the first time $(67, 75, 83, 87, 91, 99)$; 51, 107 and 115 occur in neither list.

Remark 4.4 (distance to the best known upper bound). For $n \equiv 3 \pmod 4$ the sharpest classical upper bound is Ehlich's, not the Ehlich–Wojtas bound, which governs $n \equiv 2 \pmod 4$: Ehlich's analysis bounds $\det(XX^{\mathsf T})$ by the determinant of an Ehlich-block matrix, and for the partition $n = r_1 + \cdots + r_s$ that determinant is

$$(n-3)^{n-s} \prod_{i=1}^{s} (n - 3 + 4r_i) \cdot \Big(1 - \sum_{i=1}^{s} \frac{r_i}{n - 3 + 4r_i}\Big),$$

maximized over the partition into $s = f(n)$ nearly equal parts, where $f(n) = 6$ for $11 \le n \le 59$ and $f(n) = 7$ for $n \ge 59$ [3, §6 and Thm. 25 (Satz 3.3 of Ehlich), v4]. We evaluated this in exact rational arithmetic at the three orders — parts $9, 9, 9, 8, 8, 8$ at $n = 51$; $16, 16, 15, 15, 15, 15, 15$ at 107; $17, 17, 17, 16, 16, 16, 16$ at 115 — as a check on the archived pages, which report the ratio of the *previous* record to a bound they label "Ehlich/Wojtas". The three archived ratios 0.851139, 0.840862, 0.833597 come out digit for digit, which identifies the bound in question; the new records then sit at

$$0.877569, \qquad 0.844532, \qquad 0.847607$$

of it. A gap of 12–16% therefore remains at all three orders. For comparison the Barba bound $\sqrt{2n-1}\,(n-1)^{(n-1)/2}$, which is what the survey's own table uses at $n \le 59$, gives 0.648105 for the previous $n = 51$ record — printed there as 0.6481 — and 0.668230 for the new one. This remark is a numerical computation outside the formal development; none of it is verified in Lean.

4.3. **Controls.** The certificate route is tested, not assumed.

Proposition 4.5. (1) *(Agreement with the definition.) There is an explicit* ± 1 *matrix* M_6 *of order* 6 *whose determinant, computed by expanding the Leibniz sum over all* 720 *permutations, is* -64*, and whose absolute determinant, computed through Theorem 3.1 from a Bareiss certificate, is* 64*.*

(2) *(Hypothesis (3) is necessary.) With* $L = 0$ *the hypotheses (1), (2), (4) of Theorem 3.1 hold for* every *D; explicitly they hold with* $D = 999$ *for a* 3×3 *matrix of determinant* 4*.*

(3) *(Hypothesis (2) is necessary.)* $L = I$ *is lower triangular with diagonal product* $1 \neq 0$*, and* $I \cdot (X_{51})_\sigma$ *is not upper triangular.*
(4) *(Discrimination.) At each of the three orders, hypothesis (4) fails for the value* D *altered by one unit.*
(5) *(Transcription.) The six seed strings have lengths* $25, 25, 53, 53, 57, 57$.

Moreover, at each order the determinant is not the published value increased or decreased by one normalized unit, and $|\det X|^2 \leq n^n$*, the square-free form of Hadamard's inequality.*

Item (1) is the load-bearing one: order 6 is a size at which both the definition and the certificate are available, and they return the same number. Items (2)–(4) are the statements that keep Theorem 3.1 from being vacuous or permissive. Item (5) guards the only transcription in the whole development: a dropped character in a seed would silently build a different matrix.

5. Bordering: from ±1 records to 0/1 records

The correspondence $D(n+1) = 2^n a(n)$ is classical and is the reason the two normalizations of the problem are the same problem. Here it is proved in the form that transports witnesses, for every order and with no hypothesis beyond the entry condition.

Lemma 5.1. *For every* n *and every* $n \times n$ *integer matrix* M*,*

$$\det \begin{pmatrix} 1 & \mathbf{1}^{\mathsf{T}} \\ \mathbf{1} & J - 2M \end{pmatrix} \;=\; (-2)^n \det M.$$

Consequently: every ± 1 *matrix* X *of order* $n+1$ *yields a* $0/1$ *matrix* M *of order* n *with* $|\det X| = 2^n |\det M|$*, and conversely every* $0/1$ *matrix* M *of order* n *yields a* ± 1 *matrix* X *of order* $n+1$ *with* $|\det X| = 2^n |\det M|$.

Proof. The Schur complement of the 1×1 corner is $(J - 2M) - \mathbf{1}\mathbf{1}^{\mathsf{T}} = -2M$, so the block determinant formula for a unit corner gives $\det(-2M) = (-2)^n \det M$. For the first transfer, normalize X by $\widetilde{X}_{ij} = (X_{i0} X_{00}) X_{ij} X_{0j}$; this is $\mathrm{diag}(e)\, X\, \mathrm{diag}(f)$ for sign vectors e, f, hence leaves $|\det X|$ unchanged, and it makes the first row and the first column of $\widetilde{X}$ all 1. Reindexing along the evident bijection between $\{0\} \sqcup \{1, \ldots, n\}$ and $\{0, \ldots, n\}$ exhibits $\widetilde{X}$ as the bordered matrix of the $0/1$ matrix $M_{ij} = (1 - \widetilde{X}_{i+1,j+1})/2$, and the identity applies. The converse is the same identity read in the other direction. □

Theorem 5.2. *There are explicit* $0/1$ *matrices of orders* 50*,* 106 *and* 114 *whose absolute determinants are* $N_{51} = 177761210376651936530653203125$*,* N_{107} *and* N_{115} *respectively. Hence*

$$a(50) \;\geq\; N_{51}, \qquad a(106) \;\geq\; N_{107}, \qquad a(114) \;\geq\; N_{115},$$

and $O_{51} < N_{51}$*,* $O_{107} < N_{107}$*,* $O_{115} < N_{115}$.

Proof. Apply the first transfer of Lemma 5.1 to X_{51}, X_{107}, X_{115} and cancel the factor 2^{n-1} against Theorems 4.1–4.3. The comparisons are arithmetic between explicit integers. □

The three matrices of Theorem 5.2 are not new data: each is the normalized body of the corresponding ±1 record, so the whole chain still rests on the six seed strings of Table 1. What the theorem adds is the location of these numbers in the 0/1 problem. A003432 was consulted on 29 August 2026: it carries 22 terms, $a(0)$ through $a(21)$, its b-file covers the same range, and it does not mention [1]. The three values above therefore sit at orders the sequence does not tabulate. They are lower bounds only; $a(n)$ is open for every $n \geq 22$.

Remark 5.3 (a control on the lemma). The order-7 matrix M_7 of Section 8 is $0/1$ with $\det M_7 = 32 = a(7)$. Its bordering is a ± 1 matrix of order 8 whose determinant, computed directly by expanding over all $8! = 40320$ permutations, is -4096; Lemma 5.1 predicts $(-2)^7 \cdot 32 = -4096$. The two routes agree, and $4096 = 8^4 = D(8)$, the Hadamard value at order 8.

6. Gram profiles of the three record matrices

The theory of the problem is a theory of the Gram matrix XX^{T}: Hadamard's bound is attained exactly when $XX^{\mathsf{T}} = nI$, and for $n \equiv 3 \pmod 4$ Ehlich's analysis, recalled in Remark 4.4, bounds $|\det X|$ through the shape of XX^{T}. The announcement [1] prints seeds and determinants only. The following completes that data.

Proposition 6.1. *The matrices X_{51}, X_{107}, X_{115} have XX^{T} with constant diagonal n and off-diagonal entries exactly as follows, the multiplicities summing to $n^2 - n$ in each case:*

n	*off-diagonal values with multiplicities*
51	-1 *(650)*, 1 *(1300)*, 3 *(600)*
107	-5 *(106)*, -1 *(9858)*, 3 *(1272)*, 7 *(106)*
115	-7 *(114)*, -1 *(11514)*, 3 *(1368)*, 5 *(114)*

Moreover $X_{107}X_{107}^{\mathsf{T}} = X_{107}^{\mathsf{T}}X_{107}$, while the analogous equality fails at $n = 51$ and at $n = 115$. None of the three Gram matrices is diagonal.

The values and the counts are separate statements: the counts are what makes the list exhaustive. Two observations are worth recording. The first is that exactly one of the three constructions is normal, and it is the one whose elimination needed no pivoting at all. The second is a comparison with the record it replaces: the archived page for the previous $n = 51$ record prints its Gram matrix in full, with entries $51, 3, -1, -5$, and states that the matrix is normal [7]. The new record therefore *tightens* the off-diagonal absolute values from $\{1, 3, 5\}$ to $\{1, 3\}$ and *gives up* normality. Since negating a row of X negates a row and a column of XX^{T}, the invariant to compare is the multiset of absolute values. After a suitable such normalization every off-diagonal entry of the Gram matrix of a ± 1 matrix of odd order n is congruent to n modulo 4 [5, 3]; of the three matrices above only X_{107} is already in that normalization, the entries 1 at $n = 51$ and $5, -7$ at $n = 115$ having the other sign.

7. The adjugate grand sum of a constant-row-sum matrix

We turn to the second source. Everything in this section and the next is elementary; it is included because it settles the order-7 question of [2] and because it delimits what Conjecture 1.1 can reach.

Theorem 7.1. *Let M be an $n \times n$ integer matrix with row sum k, that is $M\mathbf{1} = k\,\mathbf{1}$. Then*

$$k \cdot g(M) \;=\; n \cdot \det M.$$

Proof. Apply $\operatorname{adj} M$ to both sides of $M\mathbf{1} = k\,\mathbf{1}$ and use $(\operatorname{adj} M)M = (\det M)I$: $(\det M)\,\mathbf{1} = k\,(\operatorname{adj} M)\,\mathbf{1}$. Summing the n coordinates turns the left side into $n \det M$ and the right side into k times the sum of all entries of $\operatorname{adj} M$. □

No invertibility, no positivity and no 0/1 hypothesis is used, and the proof is valid over any commutative ring. A companion in the other direction recovers a demand vector from a single n-entry check: if $\det M \neq 0$ and $MD = (\det M)\,\mathbf{1}$ then $(\operatorname{adj} M)\mathbf{1} = D$, so $g(M) = \sum_i D_i$; this is how the numbers of Proposition 8.3 below are obtained without computing n^2 cofactors.

Corollary 7.2. *Let M have row sum k and suppose $g(M)$ and $\det M$ are coprime — the hypothesis of Conjecture 1.1. Then $g(M) \mid n$ and, if $n \neq 0$, $\det M \mid k$.*

Proof. From Theorem 7.1, $g(M)$ divides $n \det M$, and coprimality with $\det M$ leaves $g(M) \mid n$. Writing $n = g(M)c$ and cancelling $g(M)$, which is nonzero because otherwise $\det M$ would vanish and coprimality would fail, gives $k = (\det M)c$. □

Corollary 7.3. *Let M be a 7×7 integer matrix with row sum 4 whose grand sum $g(M)$ is coprime to $\det M$. Then $\det M = \pm 4$.*

The consequence for Conjecture 1.1 is the following. Let $n = 4t - 1$ be an order at which a Hadamard matrix of order $n + 1 = 4t$ exists. Every 0/1 matrix attaining $a(n)$ then has constant row sum $k = (n + 1)/2 = 2t$: bordering it produces a ± 1 matrix of order $n + 1$ attaining Hadamard's

bound, hence a Hadamard matrix, whose first row being all ones forces every other row to be balanced. By Corollary 7.2 a matrix satisfying the conjecture's hypothesis at such an order has $\det A \mid 2t$. The conjecture, even if true, can therefore never certify $K(n) \geq a(n)$ at those orders — only $K(n) \geq 2t$, which for $n = 7$ is 4 against $a(7) = 32$. We stress that this says nothing about the truth of the conjecture.

8. Order 7

The question left open by the pairing of [2] with A003432 at the first interesting order is: *does some 7×7 binary matrix of determinant $32 = a(7)$ have an odd adjugate grand sum?* That is exactly the case in which Conjecture 1.1 would apply to a maximal-determinant matrix at $n = 7$ and yield $K(7) \geq 32$. The answer is no, and the reason is the identity rather than a search.

Proposition 8.1. *Let M be a 7×7 integer matrix with row sum 4. If $\det M = 32$ then $g(M) = 56$; if $\det M = -32$ then $g(M) = -56$. In either case $g(M)$ and $\det M$ are not coprime, so M does not satisfy the hypothesis of Conjecture 1.1. The explicit matrix*

$$M_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

has row and column sums 4, $\det M_7 = 32$, $g(M_7) = 56$ and $\gcd(g(M_7), \det M_7) = 8$.

Proof. Theorem 7.1 with $n = 7$, $k = 4$ gives $4g(M) = 7 \cdot (\pm 32)$, so $g(M) = \pm 56$; and $\gcd(56, 32) = 8$. For M_7 the row and column sums and the determinant are finite evaluations, and $g(M_7) = 56$ then follows from the identity rather than from the 49 cofactors of $\operatorname{adj} M_7$. □

The hypothesis *row sum* 4 carries the classification input, and we are explicit about its status. Classically, a 7×7 binary matrix has $|\det| = 32$ if and only if its bordering is a Hadamard matrix of order 8, which forces every row sum to be 4; equivalently M is the incidence matrix of a 2-$(7, 4, 2)$ design, the complement of a Fano plane. Of this argument the bordering identity $\det H = -128 \det M$ is now Lemma 5.1 at $n = 7$, verified for every matrix; the equality case of Hadamard's inequality and the uniqueness of the order-8 Hadamard matrix up to equivalence remain classical inputs [4, 5], not part of the formal development.

The classification is independently reproduced, outside the formal development, by an exhaustive branch-and-bound search over 7×7 binary matrices with rows enumerated strictly increasing as bitmasks and Hadamard's inequality on the partial Gram matrix as the pruning bound. With no structural assumption beyond that row ordering the search visits 1 450 940 439 nodes in 15 minutes 46 seconds on one core, confirms $a(7) = 32$, and finds exactly 30 row-sorted matrices of determinant ± 32, every one of them with all row sums, all column sums equal to 4 and grand sum ± 56. This computation is not part of what is verified below; the following proposition is.

Proposition 8.2. *There is an explicit list of 30 row-sorted 0/1 matrices of order 7, each with row sum 4 and determinant ± 32, and each therefore — by Theorem 7.1, not by a second computation — with grand sum ± 56. Moreover, writing $M_{\sigma,\tau}$ for the matrix with (i, j) entry $M_{\sigma(i)\tau(j)}$, for all permutations σ, τ and every M,*

$$\det(M_{\sigma,\tau}) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \det M, \qquad g(M_\sigma) = \operatorname{sgn}(\sigma)\, g(M),$$

so $|\det M|$, $|g(M)|$ and $\gcd(g(M), \det M)$ are invariants of the row-and-column permutation class.

The 30 representatives are the complements of the 30 Fano planes on 7 labelled points, there being $5040/168 = 30$ of those, and the permutation invariance lifts them to the $30 \cdot 7! = 151\,200$ matrices of

order 7 with determinant ± 32. That these 30 are *all* the row-sorted ones is the search's claim, not a theorem here.

8.1. **The instance of** [2] **at** $n = 6$**, recomputed.** The source's own witness for $K(6) = 9$ is a 6×6 binary matrix A of determinant 9 with $B = \det(A)A^{-1}$, demand vector $D = B\mathbf{1}$ and packing $x = B^{\mathsf{T}}\mathbf{1}$. Every number in that argument is recomputed here from the matrix.

Proposition 8.3. *For the rightmost matrix A_6 of the display of* [2] *and the vectors $D = (4, 2, 5, 3, 2, 1)$, $x = (1, 2, 3, 5, 2, 4)$: $\det A_6 = 9$; $A_6 D = 9\,\mathbf{1}$, so every row is a bin filled exactly to the supply; $x^{\mathsf{T}} A_6 = 9\,\mathbf{1}^{\mathsf{T}}$, so every item is served exactly 9 times; D and x are entrywise positive with $\sum_i D_i = \sum_i x_i = 17$; the grand sum is $g(A_6) = 17$; and 17 is coprime to 9. Consequently no k with $0 < k < 9$ satisfies $9 \mid 17k$, which is the arithmetic half of $K(6) \geq 9$.*

The grand sum here is obtained from the six-entry check $A_6 D = 9\,\mathbf{1}$ through the companion of Theorem 7.1, not by computing the 36 cofactors — and is then confirmed a second time by computing them (Proposition 8.5). Note that A_6 does *not* have constant row sums: its row sums are $2, 3, 3, 3, 3, 4$. That is precisely why its grand sum escapes the divisibility that kills order 7, and it shows that the hypothesis of Theorem 7.1 is not vacuous in either direction.

8.2. **Small values of** $a(n)$**.**

Proposition 8.4. *$a(2) = 1$, $a(3) = 2$ and $a(4) = 3$, both halves; and $a(5) \geq 5$, $a(6) \geq 9$, $a(7) \geq 32$ by explicit witnesses.*

The upper halves at $n = 2, 3, 4$ are exhaustions over all 2^{n^2} binary matrices — 65536 matrices at $n = 4$, about 65 seconds — carried out over the finite type of matrices with entries in a two-element type, with a bridging lemma stating that every 0/1 integer matrix arises that way, so that the quantifier really does range over all of them. The upper halves at $n = 5, 6, 7$ are the known values of A003432 [8], reproduced by the branch-and-bound search described above; 2^{25}, 2^{36} and 2^{49} are past what the evaluator used here can enumerate, and this was priced and declined rather than attempted.

8.3. **Controls.**

Proposition 8.5. *(i) No 4×4 binary matrix has $|\det| = 4$, and $a(4) \leq 2$ is false; $a(7) \leq 31$ is false. (ii) $g(M_7) \neq 55$ and $g(M_7) \neq 57$. (iii) Computing all 36, respectively 49, cofactors of the adjugate gives $g(A_6) = 17$ and $g(M_7) = 56$, agreeing with the values Theorem 7.1 produced from a single n-entry check. (iv) A_6 has no constant row sum, and feeding the identity a wrong k at $n = 6$ gives $51 \neq 54$. (v) The leftmost matrix of the same display of* [2] *also has determinant 9, but grand sum 15, sharing the factor 3; its demand vector $(3, 3, 0, 3, 3, 3)$ has a zero entry and is therefore not a legal instance at all. (vi) The Fano incidence matrix itself has row sum 3, determinant 24 and grand sum 56, a second instance of Theorem 7.1 with a non-maximal determinant and the same grand sum.*

Item (v) explains the source's own phrase "the rightmost 6×6 matrix". Item (vi) matters because it shows $g = 56$ is not a restatement of $\det = 32$.

9. Novelty

The three numerical values of Section 4 are those of [1], and the identities of Sections 3, 5 and 7 are classical. The claim made here is narrower and is about verification. A corpus and live search conducted on 29 August 2026 — over a local mirror of the arXiv full text through 28 August 2026, the OEIS entries for A003432 and A003433, the Mathlib master branch, the Archive of Formal Proofs release of 26 August 2026, and a repository of formalized conjectures — found no maximal-determinant value verified in Lean at any order, and no verified maximal-determinant record at an order where the true value is open, in any proof assistant. One prior artifact exists and should be acknowledged: a single-commit Rocq/MathComp development of Y. Régis-Gianas [12] proves that an explicit *symmetric* ± 1 matrix of order 21 has $|\det| = 116 \cdot 20^9 = 59392000000000$, working symbolically from a Gram identity, and derives from that the maximum over symmetric matrices of order 21 — conditionally,

with the Chadjipantelis–Kounias–Moyssiadis upper bound entered as an explicit hypothesis rather than proved. Order 21 lies inside the range where the maximum is certified.

The freshness of these observations matters, so we date them. As of 29 August 2026 the e-print [1] had exactly one version, v1, submitted 23 August 2026 under math.CO and cs.DM, with no journal reference and no publisher DOI; its citation count was 0 on Semantic Scholar, OpenAlex and DataCite, and INSPIRE and Crossref held no record of it. The authors' companion repository, `Math-AI-Caltech/hadamard-maxdet` on GitHub, carried five commits and a record table with four rows: the three orders of [1] and a fourth, $n = 111$, added on 28 August 2026, after v1. Nothing in this paper concerns $n = 111$; we record it because the certificate of Section 3 is order-generic, so a further announced record is one L away from the same treatment.

The three lower bounds of Theorem 5.2 are, in addition, values of A003432 at orders that sequence does not tabulate.

10. Verification

Every numbered statement in this paper is formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; the development is free of `sorry` and declares no axiom of its own. The reasoning layer — Theorem 3.1, Lemma 5.1 with both transfers and the normalization, Theorem 7.1, Corollaries 7.2 and 7.3, Proposition 8.1, and the permutation invariance of Proposition 8.2 — depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation is exactly the finite data: for each record order, that the assembled matrix has ± 1 entries (n^2 entries), that $\prod_i L_{ii} \neq 0$, that LY vanishes below the diagonal ($n(n+1)/2$ entries) and that hypothesis (4) of Theorem 3.1 holds — about $n^3/2$ integer multiply–adds, taking 11.3, 18.3 and 29.0 seconds at $n = 51, 107, 115$; the Gram sweeps of Proposition 6.1; the exhaustions over 2^{n^2} binary matrices at $n = 2, 3, 4$ in Proposition 8.4; the order-6, order-7 and order-8 determinants and adjugates of Propositions 4.5, 8.3, 8.5 and Remark 5.3; and the 30 representatives of Proposition 8.2. Three computations are outside the formal development and are labelled as such where they occur: the branch-and-bound search that reproduces $a(7) = 32$ and the list of 30 representatives; the two-pass Bareiss elimination that produces the three certificates; and the evaluation of the Ehlich and Barba bounds in Remark 4.4. The first two are reproducible from a self-contained C program and a self-contained Python script, the third from a few lines of exact rational arithmetic; none is trusted by any theorem, since the certificate hypotheses are checked where they are used. One classical input is likewise not formalized and is flagged where it is used: that a 7×7 binary matrix of determinant ± 32 has row sum 4 (Section 8). The three record values were also reproduced from the seed strings by an independent implementation before the formal development was written, and the archived record table and the OEIS entries were read directly on the dates stated. Repository reference to be added at submission.

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