CENTRALLY SYMMETRIC BIER SPHERES ON TWELVE VERTICES, AND THE BOKOWSKI–EWALD–KLEINSCHMIDT SPHERE

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Abstract. The Bier sphere of a simplicial complex Δ on $[n]$ is the deleted join of Δ with its combinatorial Alexander dual, a simplicial $(n-2)$-sphere on at most $2n$ vertices. Holleben and Yang recently gave a purely combinatorial criterion on Δ forcing $\mathrm{Bier}(\Delta)$ to be centrally symmetric, polytopal, and yet not realizable by any centrally symmetric polytope, and exhibited one complex on $[6]$ satisfying it. We classify the whole case: the complexes on $[6]$ whose Bier spheres are the centrally symmetric neighborly 4-spheres on 12 vertices are exactly 1024 in number, they form 13 orbits under relabelling and Alexander duality, exactly 692 of them — exactly 7 of the orbits — satisfy the criterion, and the example of Holleben and Yang, read literally, is a family of 64 complexes realizing precisely those 7 orbits. All 13 orbits are polytopal, from explicit integer coordinates, so the 7 are polytopal but not centrally symmetric polytopal and the remaining 6 are the cases the criterion does not decide. We also settle the status of the sphere $\mathcal{C}$ of Bokowski, Ewald and Kleinschmidt, previously — up to two-point suspension — the only recorded simplicial sphere that is polytopal but not centrally symmetric polytopal: $\mathcal{C}$ is not the Bier sphere of any simplicial complex. Every statement this paper proves is machine-checked in Lean 4.

1. Introduction

Let Δ be an abstract simplicial complex on the ground set $[n] = \{1, \dots, n\}$: a family of subsets of $[n]$ closed under taking subsets. Following the convention we take from [4], singletons are not required to be faces. The *combinatorial Alexander dual* of Δ is

$$\Delta^* \ = \ \{\, [n] \setminus \sigma \ : \ \sigma \notin \Delta \,\},$$

again a complex on $[n]$, and the *Bier sphere* of Δ is the deleted join $\mathrm{Bier}(\Delta) = \Delta *_\Delta \Delta^*$, a simplicial complex on the doubled ground set $X_n \cup Y_n$ with $X_n = \{x_1, \dots, x_n\}$ and $Y_n = \{y_1, \dots, y_n\}$ (Section 2). Bier proved in 1992, in an unpublished manuscript recorded in [6, §5.6], that $\mathrm{Bier}(\Delta)$ is a simplicial $(n-2)$-sphere for every complex Δ with $\emptyset \neq \Delta \neq 2^{[n]}$.

Two features make these spheres interesting. First, there are very many of them: the number of combinatorially distinct Bier $(n-2)$-spheres on $2n$ vertices outgrows the number of combinatorially distinct simplicial polytopes on $2n$ vertices [3, §6.1], so asymptotically almost all of them are nonpolytopal — and yet not a single explicit nonpolytopal Bier sphere is known. Timotijević, Živaljević and Jevtić [8] pushed the polytopal range to all Bier spheres with at most 11 vertices, and Holleben and Yang [4, Thm. 1.4] to all Bier spheres with at most 12; their Problem 5.3 asks for a nonpolytopal example, and the search must now begin at 13 vertices. Second, the construction interacts with central symmetry in a controlled way. A simplicial complex with a fixed-point-free involution is *centrally symmetric*; $\mathrm{Bier}(\Delta)$ carries the candidate involution $x_i \leftrightarrow y_i$, and by [3, Prop. 6.1], quoted as [4, Prop. 3.1], $\mathrm{Bier}(\Delta)$ is centrally symmetric under it whenever $\Delta = \Delta^*$. A centrally symmetric sphere is *cs-polytopal* when it is the boundary complex of a centrally symmetric simplicial polytope in such a way that its involution is induced by $v \mapsto -v$. Every cs-polytopal sphere is polytopal, and the converse fails: before [4], and to the knowledge of its authors, exactly one counterexample was on record up to two-point suspension, the 3-sphere $\mathcal{C}$ on 10 vertices of Bokowski, Ewald and Kleinschmidt [1], reproduced as [4, Table 1].

The contribution of [4] on that second front is a finite, purely combinatorial sufficient condition — their Proposition 3.4, restated here as Proposition 2.3 — under which $\mathrm{Bier}(\Delta)$ is *not* cs-polytopal, together with one complex on $[6]$ satisfying it and a proof that its Bier sphere is nonetheless polytopal.

Their example is centrally symmetric 2-neighborly, which $\mathcal{C}$ is not, and this is what makes their Proposition 3.4, in their own words, "the first construction of cs-neighborly spheres that are polytopal but not cs-polytopal". The criterion is decidable and its inputs at $n = 6$ are complexes on six points, so the natural question is how far it reaches. That question is what this paper answers, and its answer turns out to be exact.

Results. Section 4 classifies the case $n = 6$. By [3, Prop. 6.2], quoted as [4, Prop. 3.2], the complexes Δ on [6] for which $\mathrm{Bier}(\Delta)$ is a centrally symmetric 2-neighborly 4-sphere on 12 vertices are exactly the self-dual complexes containing every subset of size at most 2: one member of each of the 10 complementary pairs of 3-subsets of [6], hence $2^{10} = 1024$ complexes. Since the dimension is $d - 1$ with $d = 5$ and $\lfloor 5/2 \rfloor = 2$, being 2-neighborly here is being neighborly. Theorem 4.3 determines the census: the 1024 complexes fall into 13 orbits under the group generated by the relabellings S_6 and Alexander duality, exactly 692 of them satisfy the criterion of [4, Prop. 3.4], those 692 form exactly 7 of the 13 orbits, and the 64 complexes that [4, Ex. 3.5] defines — its third defining clause chooses one set from each of six complementary pairs without saying which, so as written it defines 2^6 complexes rather than one — realize precisely those 7 orbits. So the published example is not merely an example: it exhausts what the criterion sees at $n = 6$.

Section 5 supplies the other half of the picture. For each of the 13 orbits we exhibit 12 points of $\mathbb{Z}^5$ realizing the corresponding Bier sphere as the boundary complex of their convex hull, verified by exact integer determinants (Theorem 5.1); combined with the census this says that the 7 orbits are polytopal but not cs-polytopal, and that the cs-polytopality of the remaining 6 is exactly what the criterion leaves open. Theorem 5.2 records the same certificate on the larger scale of [4, Thm. 1.4]: all 8072 ghost-free 12-vertex Bier spheres are polytopal, each with its facet list recomputed from its defining complex and about 3.4 million exact 6×6 integer determinants evaluated.

Section 6 settles the status of the sphere $\mathcal{C}$. Theorem 6.1 recovers from [4, Table 1] its f-vector, its non-edges and the uniqueness of its central symmetry, and then proves what is new: *$\mathcal{C}$ is not the Bier sphere of any simplicial complex*. The argument is the one [4, §2.2] suggests but does not carry out, namely that the minimal nonfaces of a Bier sphere are constrained enough to be used as an obstruction. It follows that the one previously known polytopal, non-cs-polytopal sphere is not itself a Bier sphere, so [4, Prop. 3.4] is a genuinely new construction rather than a repackaging of $\mathcal{C}$.

Sections 3 and 7 collect the supporting material: the deleted-join anchors and an exhaustive check of Bier's theorem's combinatorial shadow for every complex on $[n]$ with $n \le 5$; a correction to the ghost-vertex rule as stated in [4, §2.2], which disagrees with the membership test of $\mathrm{Bier}(\Delta)$ on 1084 of the 7581 complexes on [5]; and the arithmetic core of [4, Prop. 3.4] over $\mathbb{R}$. Section 8 describes the verification, and Section 9 the frontier.

2. Preliminaries

Throughout, a *complex on* $[n]$ is a family $\Delta \subseteq 2^{[n]}$ closed under taking subsets; $\emptyset \in \Delta$ unless Δ is the void complex $\Delta = \emptyset$, and singletons need not be faces. A subset of $[n]$ that is not a face is a *nonface*, and a minimal such is a *minimal nonface*. The two degenerate complexes are the void complex and the full simplex $2^{[n]}$.

For a face of the doubled ground set we write $A \uplus C$ for the set $\{x_i : i \in A\} \cup \{y_j : j \in C\}$, where $A, C \subseteq [n]$. For complexes K, L on $[n]$ the *deleted join* is

$$K *_\Delta L \;=\; \{\, A \uplus C \;:\; A \in K,\ C \in L,\ A \cap C = \emptyset \,\},$$

and $\mathrm{Bier}(\Delta) = \Delta *_\Delta \Delta^*$. The following unfolding of the definition is the form in which all computations below are carried out.

Lemma 2.1 (Membership test)**.** *For $A, C \subseteq [n]$,*

$$A \uplus C \in \mathrm{Bier}(\Delta) \quad\Longleftrightarrow\quad A \cap C = \emptyset, \quad A \in \Delta, \quad [n] \setminus C \notin \Delta.$$

Proof. A face of Δ^* carried to the y-labels is $[n] \setminus B$ for some $B \notin \Delta$, so a face of the deleted join is $A \uplus ([n] \setminus B)$ with $A \in \Delta$, $B \notin \Delta$ and $A \cap ([n] \setminus B) = \emptyset$, i.e. $A \subseteq B$. Putting $C = [n] \setminus B$ the last condition is $A \cap C = \emptyset$, and $B \notin \Delta$ is $[n] \setminus C \notin \Delta$. □

Two consequences are used constantly. First, $A \uplus C$ can never contain an antipodal pair $\{x_i, y_i\}$, so $\mathrm{Bier}(\Delta)$ is a subcomplex of the boundary complex $\partial\Diamond_n$ of the n-dimensional cross-polytope, whose faces are exactly the sets $A \uplus C$ with $A \cap C = \emptyset$. Second, the facets are visible:

Lemma 2.2 (Facets)**.** *The facets of* $\mathrm{Bier}(\Delta)$ *are exactly the sets* $A \uplus \big([n] \setminus (A \cup \{b\})\big)$ *with* $A \in \Delta$, $b \notin A$ *and* $A \cup \{b\} \notin \Delta$. *Each has* $n-1$ *elements, so* $\mathrm{Bier}(\Delta)$ *is pure of dimension* $n-2$*, whatever its ghost vertices.*

Proof. Immediate from Lemma 2.1: enlarging C is possible until $[n] \setminus C$ becomes a face, and $|A| + \big(n - |A| - 1\big) = n - 1$ in every case. □

A vertex of the ambient set that is not a face of Δ is called, following [4, §2.2], a *ghost vertex*. By Lemma 2.1, x_i is a vertex of $\mathrm{Bier}(\Delta)$ if and only if $\{i\} \in \Delta$ and $[n] \notin \Delta$, and y_i is a vertex if and only if $[n] \setminus \{i\} \notin \Delta$ and $\emptyset \in \Delta$; the two nondegeneracy clauses matter only for the void complex and the full simplex, where $\mathrm{Bier}(\Delta)$ is empty. A complex with no ghost vertex has $\mathrm{Bier}(\Delta)$ on the full $2n$ vertices.

We use several results of [4] as stated there. Two of them [4] quotes from [3, Props. 6.1 and 6.2], whose own name for the second property is "k-nearly neighborly": if $\Delta = \Delta^*$ then $\mathrm{Bier}(\Delta)$ is centrally symmetric, the involution being $x_i \leftrightarrow y_i$; and, for $2 \le k \le \lfloor (n-1)/2 \rfloor$, $\mathrm{Bier}(\Delta)$ is a centrally symmetric k-neighborly $(n-2)$-sphere on $2n$ vertices if and only if Δ is self-dual and contains every subset of $[n]$ of size at most k. Both sources state the first as an implication only, and that is the direction used below.

Proposition 2.3 ([4, Prop. 3.4])**.** *Let* n *be even and let* Δ *be a complex on* $[n]$ *such that* $\mathrm{Bier}(\Delta)$ *is a centrally symmetric* k*-neighborly sphere for some* $2 \le k \le (n-2)/2$. *Suppose there are distinct* $r, \ell \in [n]$ *and sets* $C, D \subseteq [n] \setminus \{r, \ell\}$ *of size* $(n-2)/2$ *with*

(1) $C \cup \{r\} \in \Delta = \Delta^*$ *and* $[n] \setminus (C \cup \{\ell\}) \in \Delta$*, and*
(2) $D \cup \{\ell\} \in \Delta = \Delta^*$ *and* $[n] \setminus (D \cup \{r\}) \in \Delta$.

Then $\mathrm{Bier}(\Delta)$ *is not cs-polytopal.*

We call the data (r, ℓ, C, D) a *certificate* for Δ, and say that Δ *satisfies the criterion* when one exists. The proof in [4] passes through McMullen's characterization of the faces of a centrally symmetric polytope in terms of a centrally symmetric transform [4, Thm. 3.3]; Section 7 isolates the part of that argument that is not a finite check.

The polytopality certificates all use the following criterion. For a set $V = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^d$ and a d-subset $F = \{i_1 < \cdots < i_d\}$ of $[m]$, write

$$D_V(F, j) \;=\; \det \begin{pmatrix} 1 & \cdots & 1 & 1 \\ v_{i_1} & \cdots & v_{i_d} & v_j \end{pmatrix}.$$

Proposition 2.4 ([4, Prop. 4.1])**.** *Let* S *be a simplicial* $(d-1)$*-sphere on* m *vertices with facet set* $\mathcal{F}(S)$*, and let* V *be* m *points of* $\mathbb{R}^d$. *If for every* $F \in \mathcal{F}(S)$ *the values* $D_V(F, j)$*,* $j \notin F$*, are all nonzero and of the same sign, then* V *realizes* S *as the boundary complex of* $\mathrm{conv}(V)$*; in particular* S *is polytopal.*

As [4] notes immediately after the statement, the companion condition on the non-facets is redundant for this direction, since condition (1) already makes S a subcomplex of $\partial\,\mathrm{conv}(V)$ and two simplicial spheres of the same dimension, one contained in the other, coincide. We use the criterion only in this direction and only over $\mathbb{Z}$: all realizations below are rational, and scaling all points by a common positive denominator D multiplies every $D_V(F, j)$ by $D^d > 0$, so no sign changes and every determinant is an exact integer determinant.

3. The construction, verified below $n = 6$

This section records the finite verifications on which everything later rests: that the encoding used computes the object Bier's theorem is about, that the deleted-join picture is the classical one, and that the ghost-vertex rule is the one derived from Lemma 2.1 rather than the one printed in [4, §2.2].

Write $M(n)$ for the number of complexes on $[n]$, the Dedekind number: $M(2) = 6$, $M(3) = 20$, $M(4) = 168$, $M(5) = 7581$, $M(6) = 7828354$ ([7, A000372]). Call a complex K on $[n]$ *sphere-like* when $\mathrm{Bier}(K)$ is pure of dimension $n-2$, every ridge of $\mathrm{Bier}(K)$ lies in exactly two facets, the dual graph on facets is connected, and $\chi(\mathrm{Bier}(K)) = 1 + (-1)^n = \chi(S^{n-2})$.

Theorem 3.1 (The combinatorial shadow of Bier's theorem below $n = 6$)**.** *For $n = 2, 3, 4, 5$, every complex on $[n]$ except the two degenerate ones is sphere-like. The numbers of sphere-like complexes are 4, 18, 166 and 7579 respectively, and the void complex and the full simplex are the two exceptions at each n.*

Proof and scope. Exhaustive computation. All $6 + 20 + 168 + 7581$ complexes on $[n]$, $n \leq 5$, are generated by deciding the subsets of $[n]$ in an order refining inclusion and admitting a subset only when all of its facets are already present, which is downward closure; the enumeration is validated twice, by reproducing the Dedekind numbers above and by testing every entry against an independent downward-closure predicate. For each complex the four conditions are evaluated on the facet list of Lemma 2.2. Both degenerate complexes are verified to be present in the enumeration and to fail, so the statement is not vacuous.

This is a shadow of Bier's theorem, not a second proof of it: a normal pseudomanifold with the Euler characteristic of a sphere need not be a sphere in dimension at least 3. What the computation certifies is that the encoding computes the object of [6, §5.6] at every instance below $n = 6$. □

Theorem 3.2 (Deleted-join anchors)**.**

(1) *$2^{[n]} *_\Delta 2^{[n]} = \partial\Diamond_n$ for $n = 1, 2, 3, 4$, and $\partial\Diamond_4$ has 81 faces, 16 facets, and is sphere-like.*

(2) *Every face of $\mathrm{Bier}(K)$ is a face of $\partial\Diamond_4$, for every one of the 168 complexes K on $[4]$.*

(3) *$\mathrm{Bier}(2^{[6]})$, formed on the ground set $[7]$, is the boundary complex of the 6-dimensional cross-polytope: it has 12 vertices, $3^6 = 729$ faces and $2^6 = 64$ facets, and equals $\partial\Diamond_6$ after deleting the two ghost labels. The complex $2^{[5]}$ on $[7]$ is a different complex whose Bier sphere again has 12 vertices, 729 faces and 64 facets.*

(4) *The Alexander dual is what closes the join up: for K the boundary of a triangle on $[3]$, the deleted join $K *_\Delta K$ has 25 faces and 6 top-dimensional faces against the cross-polytope's 8, and is not thin.*

Proof. Items (1), (3) and (4) are equalities and counts of explicitly listed face sets, checked by evaluation; item (2) is a scan over all 168 complexes on [4] and all faces of each. Item (2) also holds in general and for a one-line reason — the disjointness clause of Lemma 2.1 — but it is the finite check that is certified here. □

The ghost-vertex rule stated in [4, §2.2] reads: $\mathrm{Bier}(\Delta)$ *has a ghost vertex for every minimal nonface of size 1 or $n-1$ of Δ*. Read literally the second clause is the wrong way round. By Lemma 2.1, y_i is a vertex exactly when $[n] \setminus \{i\} \notin \Delta$, so y_i is a ghost exactly when $[n] \setminus \{i\} \in \Delta$ — the opposite of $[n] \setminus \{i\}$ being a minimal nonface.

Theorem 3.3 (The stated ghost rule and the membership test)**.** *Counting ghost vertices by minimal nonfaces of size 1 or $n-1$ disagrees with $2n$ minus the vertex count of $\mathrm{Bier}(\Delta)$ on exactly 1084 of the 7581 complexes on [5]. A witness is $\Delta = 2^{[6]}$ regarded on [7], whose only minimal nonface is $\{7\}$: the stated rule predicts one ghost and hence 13 vertices, while $\mathrm{Bier}(\Delta)$ has 12. The vertex count computed from the corrected rule agrees with the number of 1-element faces of $\mathrm{Bier}(K)$ for every one of the 168 complexes on [4]; the numbers of ghost-free complexes are 1, 64 and 6212 at $n = 3, 4, 5$, and the numbers of self-dual complexes are 4, 12 and 81, in agreement with* [7, A001206].

Proof and reading. Exhaustive computation over the enumerations of Theorem 3.1, with both rules evaluated on each complex; the witness is a single evaluation. This is a correction to a sentence, not to a result. The enumeration underlying [4, Thm. 1.4] uses the corrected rule — the companion code [5] enumerates the complexes on [6] containing no 5-subset and not containing [6], which is exactly $[6] \setminus \{i\} \notin \Delta$ for all i — and it is that reading which reproduces the count 8072 of Theorem 5.2. ☐

4. The census at $n = 6$

Fix $n = 6$ and $k = 2$. By [4, Prop. 3.2] the complexes Δ on [6] with $\mathrm{Bier}(\Delta)$ a centrally symmetric 2-neighborly 4-sphere on 12 vertices are the self-dual complexes containing every subset of size at most 2. Self-duality forces the choice on 3-subsets to be one member of each of the 10 complementary pairs $\{A, [6] \setminus A\}$ and determines the complex completely, so there are exactly $2^{10} = 1024$ of them. Write Δ_s, $s \in \{0,1\}^{10}$, for this family. Since the spheres have dimension $4 = d - 1$ with $d = 5$ and $\lfloor 5/2 \rfloor = 2$, all of them are neighborly in the sense of [4].

Two operations preserve the class and the isomorphism type of the Bier sphere: relabelling [6] by a permutation, and passing from Δ to Δ^*, which for a self-dual complex means swapping the roles of the x- and y-labels. We call two complexes *equivalent* when they are related by the group these generate, and count orbits by a canonical form.

Before the census we record the combinatorial step that makes the proof of Proposition 2.3 work, in the finite form in which it is checked.

Proposition 4.1 (Signed copies of faces)**.** *For every self-dual complex K on* [5]*, and for every one of the* 1024 *complexes Δ_s on* [6]*: whenever T is a face of the complex and $T = A \sqcup C$ is any splitting of T into two disjoint parts, $A \uplus C$ is a face of the Bier sphere. Self-duality is needed: of the* 168 *complexes on* [4]*,* 87 *fail this property, and none of the* 12 *self-dual ones does.*

Proof and reading. Exhaustive computation: for each of the 81 self-dual complexes on [5] and each of the 1024 complexes Δ_s, every face T and every subset $A \subseteq T$ are enumerated and Lemma 2.1 evaluated on $(A, T \setminus A)$; the control runs over all 168 complexes on [4]. The reason behind the computation is short — $A \subseteq T$ is a face, $C \subseteq T$ is a face, and $[n] \setminus C \notin \Delta$ is equivalent to $C \in \Delta$ by self-duality — and it is exactly the step [4] invokes as "by the first part of condition (1) and the definition of Bier spheres". What is certified here is the finite instance, on the complexes the census actually uses. ☐

Theorem 4.2 (The example of [4] is a family of 64)**.** *Clause (3) of* [4, Ex. 3.5] *prescribes one set from each of six complementary pairs without saying which, so the example as written defines $2^6 = 64$ complexes on* [6]*. All* 64 *are simplicial complexes, self-dual, centrally symmetric* 2*-neighborly, ghost free with exactly* 12 *vertices, and each Bier sphere is a pure* 4*-dimensional thin strongly connected pseudomanifold with $\chi = 2$ and exactly* 60 *facets. Every one of the* 64 *satisfies the hypothesis of Proposition 2.3 with the witness printed in* [4]*, namely $r = 3$, $\ell = 4$, $C = \{1, 2\}$, $D = \{2, 6\}$; hence by Proposition 2.3 none of the* 64 *Bier spheres is cs-polytopal. The* 64 *complexes are pairwise distinct and form* 7 *equivalence classes.*

Proof. The 64 complexes are built from the three clauses of [4, Ex. 3.5] as printed, indexed by the selector for clause (3), and every assertion is evaluated on each of the 64: the complex axioms, self-duality, the neighborliness test, the ghost-vertex count, the four combinatorial sphere conditions, the facet count, and the four membership conditions of Proposition 2.3 at the stated witness. The last sentence is a comparison of 64 canonical forms. The step from the certificate to "not cs-polytopal" is Proposition 2.3 and is cited, not reproved; see Section 7 for what part of its proof is verified here. ☐

Theorem 4.3 (The census)**.** *Let Δ_s, $s \in \{0,1\}^{10}$, be the* 1024 *complexes on* [6] *that are self-dual and contain every subset of size at most* 2*. Then:*

1. *the* 1024 *are pairwise distinct and all ghost free, and each is sphere-like in the sense of Section 3, with* $\mathrm{Bier}(\Delta_s)$ *carrying* 12 *vertices and exactly* 60 *facets; by* [4, Prop. 3.2] *each* $\mathrm{Bier}(\Delta_s)$ *is a centrally symmetric neighborly* 4*-sphere;*
2. *they form exactly* 13 *equivalence classes;*

(3) *exactly* 692 *of them satisfy the criterion of Proposition 2.3, and those* 692 *form exactly* 7 *of the* 13 *classes; the other* 332*, forming the remaining* 6 *classes, do not;*
(4) *the* 64 *complexes of Theorem 4.2 realize exactly those* 7 *classes.*

Consequently the 692 *complexes give centrally symmetric neighborly* 4*-spheres on* 12 *vertices that are not cs-polytopal, and the criterion of* [4, Prop. 3.4] *reaches, at* $n = 6$*, exactly the classes already named by* [4, Ex. 3.5].

Proof. Every clause is an exhaustive computation over the 1024 complexes, each of which is a 64-bit incidence vector over the subsets of $[6]$. For (1): the complex axioms, self-duality, containment of all ≤ 2-subsets, the ghost-vertex count, the four combinatorial sphere conditions of Section 3, the facet count, and pairwise distinctness of the 1024 codes. For (2) and (3): the canonical form of each complex is computed by minimizing its code over the 720 relabellings applied to the complex and to its Alexander dual, 1440 images in all, and the criterion of Proposition 2.3 is decided by searching all $6 \cdot 5$ ordered pairs (r, ℓ) and, for each, all pairs of 2-subsets C, D of $[6] \setminus \{r, \ell\}$ — a search of at most $30 \cdot 6 \cdot 6$ candidate certificates per complex, run to exhaustion when it fails. For (4): the multiset of canonical forms of the 692 and of the 64 complexes of Theorem 4.2 are sorted and compared, and are equal.

The final sentence is Proposition 2.3 applied to each of the 692, and is the only step that is cited rather than computed. □

Remark 4.4. The count 13 is a count of orbits of *complexes*. Equivalent complexes have isomorphic Bier spheres, so there are at most 13 isomorphism types of sphere among the 1024; that the 13 representatives give pairwise non-isomorphic spheres is not asserted here. The same caveat applies to the counts 7 and 6, and to the count 8072 in Theorem 5.2.

The smallest n at which the criterion can fire is 6, and [4, Rem. 3.6] explains why: at $n = 4$ the pairs of 2-subsets are $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$, and once C, r, ℓ are chosen the leftover pair must contain $\{r, \ell\}$, so no D exists. The computation locates the content of that remark in the D clause precisely.

Theorem 4.5 ([4, Rem. 3.6] at $n = 4$)**.** *None of the* 12 *self-dual complexes on* $[4]$ *satisfies the criterion of Proposition 2.3, while* 44 *of all* 168 *complexes on* $[4]$ *satisfy its four membership conditions once self-duality is dropped. Weakening the criterion by deleting condition (2) — keeping only* $C \cup \{r\} \in \Delta$ *and* $[4] \setminus (C \cup \{\ell\}) \in \Delta$ *— makes it fire on all* 12 *self-dual complexes on* $[4]$.

Proof. Three exhaustive searches over the 168 complexes on $[4]$, each running the certificate search to exhaustion. □

5. Polytopality

The census says which spheres are not cs-polytopal. For the conclusion "polytopal but not cs-polytopal" one needs realizations, and here we use the rational realizations produced by [4] and [5], cleared to integers as described after Proposition 2.4. The verification is independent of the source's in the one respect that matters for a mislabelled sphere: the facet lists are recomputed from the defining complexes by Lemma 2.2 and never read from the data files.

Theorem 5.1 (The thirteen classes are polytopal)**.** *For each of the* 13 *equivalence classes of Theorem 4.3 there are* 12 *points of* $\mathbb{Z}^5$ *satisfying condition (1) of Proposition 2.4 on every one of the* 60 *facets of the corresponding Bier sphere. Hence all* 13 *classes are polytopal, and the* 7 *classes satisfying the criterion are polytopal but not cs-polytopal.*

Proof. The 13 complex codes and their point sets are listed explicitly. Three things are checked. First, that the 13 listed complexes are the census: their canonical forms, sorted, equal the sorted canonical forms of the 1024 complexes of Theorem 4.3; and each is a self-dual, 2-neighborly, ghost-free complex on $[6]$ whose Bier sphere has 60 facets. Second, a transcription guard: each point set has 12 rows of 5 integers, pairwise distinct. Third, the certificate: for each sphere and each of its 60 facets, the 7 determinants $D_V(F, j)$, $j \notin F$, are computed exactly over $\mathbb{Z}$ and found nonzero and of one sign, which is $60\cdot 7 = 420$ exact 6×6 integer determinants per sphere and 5460 in all. Each facet's determinants are obtained from a single subset dynamic program giving all maximal minors of the 6×6 homogeneous coordinate matrix of the facet.

Four controls confirm that the certificate is not vacuous: it fails when two vertex labels are exchanged, when one vertex is reflected through the origin, when all twelve points are the origin, and when one sphere is given another sphere's coordinates. It does *not* fail when 1 is added to a single coordinate — the rounding of [4] leaves slack, and the first realization survives a perturbation of 50 in one coordinate, i.e. $\frac{1}{2}$ before clearing the denominator 100. That is recorded rather than replaced by a control that would break. $\square$

Theorem 5.2 (The ghost-free half of [4, Thm. 1.4])**.** *There is a list of* 8072 *pairwise distinct complexes on* [6]*, each ghost free and each with* $\mathrm{Bier}(\Delta)$ *a* 12*-vertex pure* 4*-dimensional thin strongly connected pseudomanifold with* $\chi = 2$*, and for each of them* 12 *points of* $\mathbb{Z}^5$ *satisfying condition (1) of Proposition 2.4 on every facet. All* 8072 *of these Bier spheres are therefore polytopal.*

Proof and scope. For each of the 8072 entries the defining complex is checked to be a ghost-free complex on [6], the facet list is recomputed from it by Lemma 2.2, the four combinatorial sphere conditions are evaluated, and every facet is tested against the listed points; the 8072 complex codes are checked pairwise distinct. In total about 3.4 million exact 6×6 integer determinants are evaluated, in 396 seconds and 0.82 GB of memory. A control confirms that the test is not constant: the first sphere with the second sphere's coordinates fails.

What this theorem does *not* certify is that the 8072 complexes are exhaustive, or pairwise nonisomorphic as spheres. The count itself is from [4, §4], which states that there are 8072 complexes on [6] without ghost vertices up to S_6 and Alexander duality, and adds that distinct complexes may still give isomorphic Bier spheres. A standalone C program outside the verified development reproduces that number here, reporting along the way that 7741776 of the $M(6) = 7828354$ complexes on [6] are ghost free; of those two intermediate numbers, which are ours rather than [4]'s, only $M(6)$ is independently attested [7, A000372]. The remaining 209 defining complexes behind [4, Thm. 1.4] carry ghost vertices and live on ground sets [7] through [12]; their vertex labels need a compression step that the present development does not implement, and they were verified in exact rational arithmetic outside it. $\square$

6. The Bokowski–Ewald–Kleinschmidt sphere

Let $\mathcal{C}$ be the simplicial 3-sphere on 10 vertices whose 28 facets are listed in [4, Table 1], reproducing the example of [1]. Its interest here is that it was, as [4, §3.3] records to the best of its authors' knowledge and up to two-point suspension, the only simplicial sphere known before [4] to be polytopal but not cs-polytopal, and it is natural to ask whether the construction of [4, Prop. 3.4] rediscovers it. It does not, and for a structural reason.

Theorem 6.1 ($\mathcal{C}$ is not a Bier sphere)**.** *Let* $\mathcal{C}$ *be the complex generated by the* 28 *facets of* [4, Table 1]*. Then*

(1) $\mathcal{C}$ *has* f*-vector* $(1, 10, 38, 56, 28)$ *and is a pure* 3*-dimensional thin strongly connected pseudomanifold with* $\chi = 0 = \chi(S^3)$*;*
(2) $\mathcal{C}$ *has exactly* 7 *non-edges: the five pairs* $\{1, 10\}$*,* $\{2, 5\}$*,* $\{3, 6\}$*,* $\{4, 9\}$*,* $\{7, 8\}$ *of the involution* φ *of* [1]*, together with the two exceptional pairs* $\{1, 9\}$ *and* $\{4, 10\}$*;*
(3) *of the* 945 *perfect matchings of the vertex set, exactly* 2 *induce automorphisms of* $\mathcal{C}$*, and exactly one of those induces a fixed-point-free involution on the nonempty faces, namely* φ*; the other is* $1 \leftrightarrow 4$*,* $2 \leftrightarrow 3$*,* $5 \leftrightarrow 6$*,* $7 \leftrightarrow 8$*,* $9 \leftrightarrow 10$*, and it fixes a face setwise;*
(4) $\mathcal{C}$ *is not the Bier sphere of any simplicial complex.*

Proof. Items (1)–(3) are computations on the transcribed facet list: the face poset is generated from the 28 facets and the f-vector, Euler characteristic, thinness and dual-graph connectivity read off it; the non-edges are the 7 pairs among the 45 that are not faces; and all 945 perfect matchings of the 10 vertices are enumerated, each tested for sending facets to facets and for fixing no nonempty face setwise. The transcription is self-checking, in that (1) and (3) reproduce the f-vector and the involution as printed in [4, §3.3, Table 1].

Item (4) is the new one, and it is a finite search of 64 cases. Suppose $\mathcal{C} \cong \mathrm{Bier}(K)$ for a complex K on $[n]$. By Lemma 2.2, $\mathrm{Bier}(K)$ is pure of dimension $n-2$ whatever its ghost vertices, so $n-2=3$, i.e. $n=5$; then $\mathrm{Bier}(K)$ has at most 10 vertices and $\mathcal{C}$ has exactly 10, so K has no ghost vertex and the isomorphism matches $X_5 \cup Y_5$ with the vertex set of $\mathcal{C}$. By Lemma 2.1 no antipodal pair $\{x_i, y_i\}$ is a face, so the induced partition into x- and y-labels is a perfect matching of the vertices of $\mathcal{C}$ contained in its non-edge graph. By item (2) that graph has 7 edges, and exactly 2 of its subsets are perfect matchings. Fixing one of the two, and fixing which element of each pair carries the x-label (2^5 choices), determines K as the induced subcomplex of $\mathcal{C}$ on the x-side. All $2 \cdot 2^5 = 64$ resulting complexes are formed and, for each, the full membership table of $\mathrm{Bier}(K)$ over all $32 \cdot 32$ pairs (A, C) is compared with the face table of $\mathcal{C}$; none agrees.

Item (4) is a negative result, so we record what its non-vacuity rests on and what it does not. Each half of the comparison is exercised in both directions elsewhere in the development: the membership test of Lemma 2.1 is the one that returns *true* on every splitting of every face in Proposition 4.1, and that returns *false* on some splitting for 87 of the 168 complexes on [4] there; and the face table of $\mathcal{C}$ is the one from which items (1) and (2) recompute the printed f-vector and the 7 non-edges. The composite 64-case procedure, however, is never run to a positive answer. On the nearest genuine Bier sphere, $\mathrm{Bier}(K)$ for $K = \{A \subseteq [5] : |A| \le 2\}$ — self-dual, hence a centrally symmetric 3-sphere candidate on 10 vertices — what is checked is the vertex count and the four combinatorial sphere conditions of Section 3, not that the search recognizes it as a Bier sphere. □

Remark 6.2. Item (4) is the use [4, §2.2] proposes for its classification of the minimal nonfaces of a Bier sphere — "we can also use the list of nonfaces above to prove that a certain sphere is not the Bier sphere of any simplicial complex" — applied to $\mathcal{C}$, which [4] does not do. [4, §3.3] separates its construction from $\mathcal{C}$ by a weaker observation, that $\mathcal{C}$ is not centrally symmetric k-neighborly for any $k \ge 2$ because of its non-edges $\{1, 9\}$ and $\{4, 10\}$, which join vertices that are not antipodal.

7. The arithmetic core of the obstruction

The proof of Proposition 2.3 in [4] assumes $\mathrm{Bier}(\Delta)$ is the boundary complex of a centrally symmetric polytope P, takes a centrally symmetric transform of the vertex set of P — for a centrally symmetric $(n-1)$-polytope with $2n$ vertices this is a centrally symmetric set of $2n$ reals $\pm\bar{z}_1, \dots, \pm\bar{z}_n$ — and applies McMullen's criterion [4, Thm. 3.3]: a set of r points with distinct suffixes spans a face of P if and only if the corresponding signed sum of transform values lies in the relative interior of the convex hull of the signed sums over the complementary indices. In dimension one that hull is an interval whose endpoints are $\pm \sum_{j \notin T} |\bar{z}_j|$, and the four faces produced by conditions (1) and (2) then give four strict inequalities which are jointly contradictory.

Theorem 7.1 (The arithmetic core)**.** *Let z assign a real number to each index, and let S be a finite set of indices.*

(1) *For every sign vector $\varepsilon \in \{\pm 1\}^S$ one has $\sum_{j \in S} \varepsilon_j z_j \le \sum_{j \in S} |z_j|$, and the bound is attained by the sign vector of z; if $z_j > 0$ for all $j \in S$ then $\sum_{j \in S} |z_j| = \sum_{j \in S} z_j$.*

(2) *The four inequalities that conditions (1) and (2) of Proposition 2.3 produce are inconsistent: for any two indices r, ℓ and any finite index sets C, D, J, J', no real values make*

$$\sum_{i \in C} z_i + z_r < \sum_{j \in J} z_j + z_\ell, \qquad \sum_{j \in J} z_j + z_r < \sum_{i \in C} z_i + z_\ell,$$
$$\sum_{i \in D} z_i + z_\ell < \sum_{j \in J'} z_j + z_r, \qquad \sum_{j \in J'} z_j + z_\ell < \sum_{i \in D} z_i + z_r$$

all hold at once. The first two force $z_r < z_\ell$ and the last two force $z_\ell < z_r$.

All four inequalities are needed, and their strictness is needed: any three of them are satisfiable, and replacing all four by their non-strict versions makes them consistent.

Proof and scope. Part (1) is a termwise comparison and an explicit sign vector; part (2) is linear arithmetic on the six quantities $\sum_C z$, $\sum_J z$, $\sum_D z$, $\sum_{J'} z$, z_r, z_ℓ, with no relation between the index sets required. The controls are explicit witnesses — for instance $(0, 0, 0, 2, 0, 1)$ satisfies the first three inequalities, and the all-zero assignment satisfies all four non-strict ones.

What is *not* proved here, and is cited from [4] and its sources, is McMullen's criterion [4, Thm. 3.3] itself, the existence of a centrally symmetric transform, and the identification of $\operatorname{relint}\operatorname{conv} S$ with the open interval $(\inf S, \sup S)$ for a finite $S \subseteq \mathbb{R}$. So the division of labour in this paper is: the certificates of Section 4 are verified, the passage from McMullen's criterion to the contradiction is verified, and McMullen's criterion is taken from the literature. □

8. Verification

The ten theorems of this paper and Proposition 4.1 have been formally verified in Lean 4 against *Mathlib*; the development is free of `sorry`, and the repository reference is to be added at submission. Lemmas 2.1 and 2.2 are the definitional unfoldings the development is built on rather than statements inside it, and Propositions 2.3 and 2.4 are quoted from [4] and not reproved here. Where the text quotes a number that no theorem here covers — the exhaustiveness of the 8072 and the enumeration counts around it in Theorem 5.2, and the 13-vertex counts of Section 9 — it says so at that point. The finite content is carried by compiled evaluation through Lean's `native_decide`, which is exactly what the exhaustive searches need: the enumerations of all 6, 20, 168 and 7581 complexes on $[n]$, $n \leq 5$ (Theorems 3.1, 3.2, 3.3, 4.5, Proposition 4.1); the 1024-complex census and its 1440-image canonical form (Theorems 4.2, 4.3); the 5460 and the roughly 3.4 million exact integer determinants of Theorems 5.1 and 5.2; and the 945 matchings and 64 candidate splittings of Theorem 6.1. Those statements depend on `propext`, `Quot.sound`, sometimes `Classical.choice`, and the per-declaration axiom that `native_decide` introduces. Theorem 7.1 is the exception: it is proved over $\mathbb{R}$ in the ordinary way and depends on no evaluation axiom. Whole-development time is under 14 minutes on one machine, dominated by Theorem 5.2. Two independent external checks validate the encoding of complexes and of Alexander duality rather than assuming it: the enumerator reproduces the Dedekind numbers $6, 20, 168, 7581$ [7, A000372], and the self-dual counts $4, 12, 81$ match [7, A001206]. The realization data of [5] was re-verified in exact rational arithmetic outside the formal development before any of it was transcribed, with the Bier facet lists recomputed and compared against the published ones for all 8072 ghost-free spheres, agreeing in every case.

9. Outlook

Two questions are left open by the census. The first is local: the 6 classes of Theorem 4.3 that the criterion does not reach. Their Bier spheres are centrally symmetric neighborly 4-spheres on 12 vertices, they are polytopal by Theorem 5.1, and whether any of them is cs-polytopal is decided neither here nor, as far as we could determine, elsewhere. A cs realization would be 6 points of $\mathbb{R}^5$ together with their negatives, so this is a 30-parameter search, half the size of the searches that produced the realizations used above.

The second is [4, Prob. 5.3], an explicit nonpolytopal Bier sphere. With 12 vertices settled the next case is 13, which by the parity of $2n - g$ forces an odd number g of ghost vertices; the dominant part is $g = 1$ on the ground set [7], where the data is a complex on the six non-ghost elements containing all six singletons and not the six-set itself. A standalone C enumeration outside the formal development — covered by no theorem above, and so an unverified computation — counts 7785061 such complexes and 16142 classes under S_7 and Alexander duality, roughly twice the 8281 defining complexes already settled at 12 vertices; the optimization of [4, §4] applies to them unchanged. But a failure to find a realization is not a theorem, and it is worth saying why no finite certificate is presently available for the negative side. The finite nonrealizability certificates for this kind of object — final polynomials in the sense of [2] — live at the level of the chirotope and refute realizability of an oriented matroid. [4, §5] reports having searched for a Bier sphere that is not even oriented-matroid realizable and having failed, and asks in its Question 5.4 whether all Bier spheres are oriented-matroid realizable. A positive answer there would mean that no chirotope-level certificate can ever exist for a Bier sphere, and a nonpolytopal example would have to be separated from the realizable oriented matroids by an argument about its realization space. On that reading Question 5.4, and not Problem 5.3, is the question on which the finiteness of the whole problem turns — and it too is empty below 13 vertices, since everything below is polytopal.

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