# THE FINITE STEP OF THE $\mathcal{B}$-FREE GRAPHS CONJECTURE, WITHOUT CORONA HYPOTHESES

KOREA SUPERINTELLIGENCE LABS

Abstract. Litjens, Polak and Sivaraman call a graph sum-perfect when every induced subgraph $H$ satisfies $\alpha(H)+\omega(H)\geq|V(H)|$, characterise the class by twenty-seven forbidden induced subgraphs, and conjecture that dropping three of them — leaving the class $\mathcal{B}$ of the twelve six-vertex bipartite graphs with a perfect matching together with their twelve complements — gives $\alpha(G)+\omega(G)\geq|V(G)|-1$ for every $\mathcal{B}$-free $G$. A recent preprint of Frijio reduces the conjecture, by structural arguments, to a finite exclusion: twelve parameter triples $(n,a,w)$ with $n=a+w+2\leq 14$, each of which must be shown not to occur. That exclusion is the whole computational core of the proof, and the evidence offered for it is a floating-point mixed-integer model and a bespoke solver that print `UNSAT` without a certificate.

We prove the exclusion from strictly weaker hypotheses. For each of the twelve triples we show that no $\mathcal{B}$-free graph on $n$ vertices has $\alpha=a$ and $\omega=w$, using none of the corona constraints that the reduction supplies and covering both values of the intersection parameter $t=|S\cap K|$, where the source's table forms only twenty-one of the twenty-four cases. That the corona constraints are dispensable was measured, not assumed: each remaining hypothesis is shown load-bearing by an explicit seven-vertex graph. We also certify the encoding step the source reports but does not check — on all $2^{15}$ labelled six-vertex graphs, the inequality form of $\mathcal{B}$-membership agrees with "bipartite with a perfect matching or the complement of one" — and refine the resulting count 4400 into $2200+2200$ with the two halves disjoint. Finally we record the conjecture itself as a consequence of the twelve exclusions and one named hypothesis, the structural reduction, which is stated but not proved here. Every statement is machine-checked in Lean 4; the twenty-four infeasibility claims are discharged against stored LRAT certificates by a checker that is itself verified in Lean, so no property of the SAT solver is assumed.

## 1. Introduction

1.1. **Sum-perfect graphs and the class $\mathcal{B}$.** All graphs are finite, simple and undirected. For a graph $G$ we write $\alpha(G)$ for the largest size of a stable (independent) set and $\omega(G)$ for the largest size of a clique. Litjens, Polak and Sivaraman [1] call $G$ *sum-perfect* if every induced subgraph $H$ of $G$ satisfies $\alpha(H)+\omega(H)\geq|V(H)|$, and characterise sum-perfect graphs by a list $\mathcal{F}$ of twenty-seven forbidden induced subgraphs. Twenty-four of the twenty-seven — their $H_2,\dots,H_{25}$ — form the class

$$\mathcal{B}\;=\;\{\,H:|V(H)|=6,\ H\text{ bipartite with a perfect matching}\,\}\;\cup\;\{\,\overline{H}:H\text{ in the first set}\,\},\tag{1}$$

twelve isomorphism types and their twelve complements. Forbidding only these twenty-four enlarges the class, and [1, Conjecture 3.1] asserts that the conclusion then weakens by exactly one:

**Conjecture 1.1** (Litjens–Polak–Sivaraman)**.** Every $\mathcal{B}$-free graph $G$ satisfies $\alpha(G)+\omega(G)\geq|V(G)|-1$.

The published evidence for Conjecture 1.1 is a single clause: it "was verified by computer to be true for all graphs with at most 10 vertices" [1]. The bound is sharp: $C_5$ is $\mathcal{B}$-free with $\alpha+\omega=n-1$, which is precisely why $C_5$ stays in $\mathcal{F}$ and outside $\mathcal{B}$ (Proposition 7.1).

1.2. **The reduction and its finite step.** A recent preprint of Frijio [2] proves Conjecture 1.1. Its Sections 2 and 3 are structural: a Hall-type exchange argument, a corona matching proposition drawn there from the set-and-collection lemma of Levit and Mandrescu [3], and two double counts cut

a hypothetical smallest counterexample down to twelve parameter triples $(n, a, w)$ with $n = a+w+2$ and $n \le 14$ — its Table 1, reproduced here as (2). Section 4 turns twenty-one quadruples $(n, a, w, t)$, where $t = |S \cap K|$ for a maximum stable set $S$ and a maximum clique $K$, into binary feasibility systems and reports each infeasible.

That report is the whole computational core of the proof, and it is not independently checkable. The ancillary material of [2] contains a SciPy/HiGHS floating-point branch-and-bound model and a hand-written CDCL solver of about five hundred lines which prints `UNSAT` and a conflict count and emits no DRAT or LRAT proof. Neither produces an artefact a third party can verify, and the systems are large enough — up to 203 variables and 368 477 clauses in the source's own encoding — that reading the solver is not a substitute.

1.3. **What is settled here.** We prove the finite step, in a form strictly stronger than the one the reduction needs, and in a form a reader can check without trusting any solver.

- **The exclusion, corona-free and for both $t$.** For each of the twelve triples of (2) and each $t \in \{0, 1\}$, the finite object we call a *cell* (Definition 2.3) does not exist; consequently no $\mathcal{B}$-free graph on $n$ vertices has $\alpha = a$ and $\omega = w$ (Theorem 5.2). A cell carries only $\mathcal{B}$-freeness, the stable set, the clique, and the two maximality bounds $\alpha \le a$, $\omega \le w$. It carries none of the corona constraints that [2, Prop. 2.3] supplies to its systems, and the exclusion covers the three quadruples $(13, 6, 5, 1)$, $(14, 6, 6, 1)$, $(14, 7, 5, 1)$ that the source's table never forms.
- **Measured, not assumed.** Every remaining hypothesis of a cell is load-bearing: drop $\alpha \le a$, or $\omega \le w$, or $\mathcal{B}$-freeness, and an explicit seven-vertex graph survives (Proposition 7.1). The corona constraints were removed one family at a time from systems that had first been solved with them (Remark 5.4).
- **The encoding step, certified.** The source states $\mathcal{B}$-membership by an inequality pattern rather than by isomorphism types, and reports — without certifying — that the two agree on all $2^{15}$ labelled six-vertex graphs. We prove that agreement (Theorem 3.1) and refine the resulting count 4400 into $2200 + 2200$ with the two halves disjoint (Theorem 3.2).
- **The conjecture, conditionally.** Conjecture 1.1 follows from the twelve exclusions together with one hypothesis, namely the structural reduction of [2, §§2–3], which is *not* proved here. We state that hypothesis explicitly and derive the conjecture from it (Theorem 6.1), so that the boundary between what is verified and what is assumed is a quantified statement rather than a remark.

The twelve triples are

$$(n, a, w) \in \{(7,3,2),\ (8,3,3),\ (8,4,2),\ (9,4,3),\ (10,4,4),\ (10,5,3),\ (11,5,4),\ (12,5,5),\ (12,6,4),\ (13,6,5),\ (14,6,6),\ (14,7,5)\}, \tag{2}$$

all satisfying $n = a + w + 2$.

1.4. **Method.** Each of the twenty-four exclusions is a propositional refutation. We build, from the definition of a cell, a CNF formula $\Phi(n, a, w, t)$ in $\binom{n}{2}$ variables, one per unordered pair of positions, whose satisfying assignments include every suitably enumerated cell (Proposition 4.2); a symmetry-breaking argument (Proposition 4.3) shows the enumeration may always be normalised, so unsatisfiability of $\Phi$ empties the cell outright. The refutations are produced by CaDiCaL 2.1.2 [4] and stored as LRAT proofs [5], which are then replayed by a checker verified inside the proof assistant. Only the direction *cell* $\Rightarrow$ *satisfying assignment* is proved: a clause we forgot to emit can only make the solver's task harder, never make a refutation unsound.

## 2. Preliminaries

2.1. **Graphs on $[n]$.** Write $[n] = \{0, 1, \dots, n-1\}$. A graph on $[n]$ is given by a symmetric adjacency predicate on $[n]$; all statements below quantify over distinct vertices, so loops play no role. A set

$S \subseteq [n]$ is *stable* if no two distinct members are adjacent and a *clique* if every two distinct members are adjacent; $\alpha(G)$ and $\omega(G)$ are the largest sizes of such sets. For $a, w \in \mathbb{N}$ we write $\alpha(G) = a$ for the conjunction of "some stable set has $a$ elements" and "every stable set has at most $a$ elements", and likewise for $\omega$.

2.2. **$\mathcal{B}$-freeness as a pattern.** Definition (1) describes $\mathcal{B}$ by isomorphism types. For the purpose of a propositional encoding it is more convenient, and it is what [2] does, to describe $\mathcal{B}$-freeness by two patterns on six labelled vertices.

**Definition 2.1.** A graph $G$ *contains the $\mathcal{B}$-pattern* if there are six distinct vertices $a_1, a_2, a_3, b_1, b_2, b_3$ of $G$ such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are both stable and $a_i b_i$ is an edge for $i = 1, 2, 3$. It *contains the co-$\mathcal{B}$-pattern* if the same holds with "stable" replaced by "clique" and "$a_i b_i$ is an edge" replaced by "$a_i b_i$ is a non-edge". $G$ is *$\mathcal{B}$-free* if it contains neither pattern.

Complementation exchanges the two patterns, so the class of $\mathcal{B}$-free graphs is self-complementary, as (1) makes it. That Definition 2.1 really describes the class (1) is the content of Section 3; it is the step [2] reports as a computation and does not certify, and without it the entire development would rest on an unchecked reading of one sentence.

2.3. **Cells.** The finite objects the exclusion is about are the following.

**Lemma 2.2.** *If $S$ is a stable set and $K$ a clique of a graph, then $|S \cap K| \leq 1$.*

*Proof.* Two distinct vertices of $S \cap K$ would be simultaneously adjacent and non-adjacent. □

Lemma 2.2 is the reason the intersection parameter $t$ below ranges over $\{0, 1\}$ and over nothing else.

**Definition 2.3.** Let $n, a, w, t \in \mathbb{N}$. A *cell of type* $(n, a, w, t)$ is a pair $(G, \varphi)$ where $G$ is a graph on $[n]$ and $\varphi$ is an enumeration of $[n]$ — a bijection from positions to vertices — such that

(C1) the positions $[0, a)$ carry a stable set of $G$;
(C2) the positions $[a - t,\, a - t + w)$ carry a clique of $G$;
(C3) $\alpha(G) \leq a$;
(C4) $\omega(G) \leq w$;
(C5) $G$ is $\mathcal{B}$-free.

We write $\mathrm{Cell}(n, a, w, t)$ for the set of cells of type $(n, a, w, t)$.

For $t \leq \min(a, w)$ the two position blocks overlap in exactly $t$ positions, so a cell is a $\mathcal{B}$-free graph carrying a stable set of size $a$ and a clique of size $w$ that meet in $t$ vertices, with no larger stable set and no larger clique. Throughout, $n = a + w + 2$, so the four blocks

$$S \setminus K = [0, a - t), \quad S \cap K = [a - t, a), \quad K \setminus S = [a, a - t + w), \quad R = [a - t + w, n)$$

have sizes $a - t$, $t$, $w - t$ and $t + 2$ respectively.

The enumeration is carried explicitly because the refutation to come is a statement about a labelled formula; Proposition 4.4 removes it again.

*Remark* 2.4. What Definition 2.3 omits is as important as what it contains. The systems of [2, §4] carry, besides the analogues of (C1)–(C5), a block of corona variables and constraints — its equations (4.5)–(4.10) — supplied by [2, Prop. 2.3]. That proposition asserts that a maximum stable set $S$ can be matched by edges to at least $\alpha(G)$ vertices outside $S$, each with at most two neighbours in $S$, and dually for a maximum clique; the source derives it from the set-and-collection lemma of Levit and Mandrescu [3], which supplies a matching from $S \setminus \mathrm{core}(G)$ into $\mathrm{corona}(G) \setminus S$, where $\mathrm{core}(G)$ and $\mathrm{corona}(G)$ are respectively the intersection and the union of all maximum stable sets. A cell carries none of that block. Every exclusion below is therefore an exclusion from a strictly weaker hypothesis than the source's Proposition 4.1 asserts.

All numbering of [2] used in this paper — Theorem 1.1, Proposition 2.3, Lemma 3.2, Table 1, Proposition 4.1, and the equations (4.1)–(4.10) — is that of the e-print, which as of 29 August 2026 stands at v1 and has had no later version.

## 3. The class $\mathcal{B}$ on six vertices

Fix the vertex set $\{0,1,\dots,5\}$; there are $2^{15} = 32768$ labelled graphs on it. Three Boolean predicates are needed. *Bipartiteness* is tested as "some two-colouring of the six vertices leaves no edge monochromatic", a search over $2^6 = 64$ colourings; *having a perfect matching* as "some ordering $v_1 \dots v_6$ has $v_1v_2$, $v_3v_4$, $v_5v_6$ all edges", a search over 720 orderings; and the $\mathcal{B}$-pattern of Definition 2.1 as "some ordering $a_1a_2a_3b_1b_2b_3$ makes both triples stable with $a_ib_i$ edges", again 720 orderings.

**Theorem 3.1.** *Let $G$ be a graph on $\{0,\dots,5\}$. Then $G$ contains the $\mathcal{B}$-pattern if and only if $G$ is bipartite and has a perfect matching, and $G$ contains the co-$\mathcal{B}$-pattern if and only if $\overline{G}$ is bipartite and has a perfect matching. Consequently, on six vertices Definition 2.1 agrees with membership in the class $\mathcal{B}$ of* (1)*.*

*Proof sketch.* Both equivalences are established by *exhaustive computation* over all $2^{15} = 32768$ labelled graphs on $\{0,\dots,5\}$: the three predicates above are evaluated on each graph and on its complement, and the list of graphs at which any of the two equivalences fails is computed and found to be empty. What the computation does not do is relate the Boolean pattern test to the quantified form of Definition 2.1; that bridge is a separate, purely logical argument — a witnessing ordering of six distinct vertices from $\{0,\dots,5\}$ is a permutation of $\{0,\dots,5\}$, and conversely — and it is proved without computation. The mathematical content of the computation is the classical observation that a perfect matching of a bipartite graph on six vertices forces the two sides to have three vertices each, so that a member of $\mathcal{B}$ is exactly a pair of stable triples joined by a perfect matching. □

**Theorem 3.2.** *Of the* 32768 *graphs on the labelled vertex set $\{0,\dots,5\}$, exactly* 2200 *are bipartite with a perfect matching and exactly* 2200 *have complement bipartite with a perfect matching. No graph is both. Hence exactly* 4400 *of them lie in $\mathcal{B}$.*

*Proof sketch.* One pass over the 32768 graphs, accumulating the three tallies $|\mathcal{B}_{\mathrm{bip}}|$, $|\mathcal{B}_{\mathrm{co}}|$ and $|\mathcal{B}_{\mathrm{bip}} \cup \mathcal{B}_{\mathrm{co}}|$ simultaneously; the result is $(2200, 2200, 4400)$, and $2200 + 2200 = 4400$ is the disjointness. Exhaustive computation. □

The total 4400 is the figure [2] reports from its own encoding check; the split into two disjoint halves of 2200 is not recorded there. The count is consistent with (1). Litjens, Polak and Sivaraman record that "there are exactly 12 bipartite graphs on 6 vertices containing a perfect matching" — their $H_2, \dots, H_{13}$, with complements $H_{14}, \dots, H_{25}$ (quoted from the e-print `arXiv:1710.07546v1` of [1]) — so those twelve isomorphism types account for the first 2200 labellings, and their complements for the other 2200.

Two further facts about $\mathcal{B}$ are worth recording, because they are used as controls in Section 7.

**Proposition 3.3.** *Membership of the $\mathcal{B}$-pattern in a graph on $[n]$ is decided by the complete search over all $n^6$ ordered six-tuples of vertices, and the corresponding search on the complement decides the co-$\mathcal{B}$-pattern. Moreover every graph on at most five vertices is $\mathcal{B}$-free.*

*Proof.* The pattern is by definition the existence of an ordered six-tuple of distinct vertices with prescribed adjacencies, so the search is complete by inspection. A witness needs six distinct vertices, which a graph on at most five does not have. □

## 4. The formula, and why a refutation suffices

4.1. **The variables and the seven clause families.** Fix $(n, a, w, t)$ with $n = a + w + 2$, $t \le 1$, $a \ge 1$, $w \ge 1$. The formula $\Phi(n, a, w, t)$ has one variable $x_{ij}$ for each unordered pair $\{i, j\}$ of *positions*

in $[n]$, meaning "the vertices at positions $i$ and $j$ are adjacent" — $\binom{n}{2}$ variables, from 21 at $n = 7$ to 91 at $n = 14$. Its clauses fall into seven families:

| family | clauses | count |
|---|---|---|
| fixS | $\neg x_{ij}$ for $0 \le i < j < a$ | $\binom{a}{2}$ |
| fixK | $x_{ij}$ for $a - t \le i < j < a - t + w$ | $\binom{w}{2}$ |
| alpha | $\bigvee_{i<j\in U} x_{ij}$, one per $(a{+}1)$-set $U \subseteq [n]$ | $\binom{n}{a+1}$ |
| omega | $\bigvee_{i<j\in U} \neg x_{ij}$, one per $(w{+}1)$-set $U \subseteq [n]$ | $\binom{n}{w+1}$ |
| bfree | two per pair of disjoint triples and matching between them | $6\binom{n}{3}\binom{n-3}{3}$ |
| kLex | the $K \setminus S$ rows over $R$ are non-decreasing | $(w{-}t{-}1)(2^{\,t+2} - 1)$ |
| sLex | the $S \setminus K$ rows over $[a, n)$ are non-decreasing | $(a{-}t{-}1)(2^{\,w+2} - 1)$ |

The alpha family says every $a+1$ positions span an edge, i.e. $\alpha \le a$; the omega family says every $w+1$ positions miss an edge, i.e. $\omega \le w$; the bfree family forbids both patterns of Definition 2.1 at every unordered pair $\{A, B\}$ of disjoint three-sets of positions and each of the six bijections $A \to B$. The two lexicographic families are symmetry breaking and are discussed in Section 4.2; the lengths $|R| = t + 2$ and $|[a, n)| = w + 2$ are what make their clause counts small. The totals run from 947 clauses at $(7, 3, 2, 0)$ to 368 544 at $(14, 6, 6, 0)$; the full list is in Table 1.

4.2. **Sorted cells and the lexicographic families.** For a cell $(G, \varphi)$ and a list of columns $c_1 < \cdots < c_m$, the *row* of position $i$ over those columns is the binary numeral whose $k$-th bit, most significant first, records whether $\varphi(i)$ and $\varphi(c_k)$ are adjacent.

**Definition 4.1.** A cell $(G, \varphi)$ of type $(n, a, w, t)$ is *sorted* if the rows of the positions $0, 1, \ldots, a-t-1$ over the columns $[a, n)$ are non-decreasing, and the rows of the positions $a, a+1, \ldots, a+(w-t)-1$ over the columns $R = [a - t + w, n)$ are non-decreasing.

**Proposition 4.2.** *Let $n = a + w + 2$, $t \le 1$, $a \ge 1$ and $w \ge 1$. If $(G, \varphi)$ is a sorted cell of type $(n, a, w, t)$, then the assignment $x_{ij} := [\,\varphi(i)\varphi(j) \in E(G)\,]$ satisfies $\Phi(n, a, w, t)$. Consequently, if $\Phi(n, a, w, t)$ is unsatisfiable then there is no sorted cell of that type.*

*Proof sketch.* One lemma per clause family, each a direct translation of the corresponding cell condition; the alpha and omega families use that the positions of a violating set are distinct, and the bfree family uses that a violated clause exhibits exactly the six vertices and the matching of Definition 2.1. The two lexicographic families are the ones that need Definition 4.1: they are *false* for a badly enumerated cell. No computation enters this proposition. Only this direction — cell ⇒ satisfying assignment — is proved anywhere in this paper; the converse is never used, which is why an omitted clause could weaken the result but could not invalidate it. □

**Proposition 4.3.** *Let $n = a + w + 2$, $t \le 1$, $a \ge 1$, $w \ge 1$. Every cell of type $(n, a, w, t)$ can be re-enumerated — the graph unchanged, only $\varphi$ replaced — to a sorted cell of the same type. Consequently, if $\Phi(n, a, w, t)$ is unsatisfiable then $\mathrm{Cell}(n, a, w, t) = \emptyset$.*

*Proof sketch.* Permuting the positions of one block permutes exactly the rows of the corresponding submatrix of the adjacency matrix and changes nothing else: the positions inside $S$ are pairwise non-adjacent and those inside $K$ pairwise adjacent by (C1) and (C2), and every other cell condition — (C3), (C4), (C5) — is a statement about $G$ alone, with no mention of $\varphi$. The two breaks are compatible because they are nested. The $K \setminus S$ rows are read over $R$ only, a set of columns that no reordering of $S \setminus K$ touches; so sort $K \setminus S$ first, by that key, and then sort $S \setminus K$ by its full rows over $[a, n)$. The second sort fixes $R$ pointwise, so it leaves the first key, and hence the first sort, standing. Each sort is the existence of a permutation of a finite list of naturals. No computation enters this proposition. □

**Proposition 4.4.** *Let $G$ be a $\mathcal{B}$-free graph on $[n]$ carrying a stable set $S$ with $|S| = a$ and a clique $K$ with $|K| = w$, and suppose $\alpha(G) \le a$ and $\omega(G) \le w$. Then $\mathrm{Cell}(n, a, w, |S \cap K|) \ne \emptyset$. Consequently, if $\mathrm{Cell}(n, a, w, 0) = \mathrm{Cell}(n, a, w, 1) = \emptyset$, then no $\mathcal{B}$-free graph on $n$ vertices has $\alpha = a$ and $\omega = w$.*

*Proof sketch.* List $S \setminus K$, then $S \cap K$, then $K \setminus S$, then the remaining vertices, each block in increasing order; this enumeration puts $S$ at the positions $[0, a)$ and $K$ at the positions $[a - t, a - t + w)$ with $t = |S \cap K|$, and (C3)–(C5) are inherited unchanged. The second sentence follows from the first with Lemma 2.2, which restricts $t$ to $\{0, 1\}$. Bookkeeping about where the four blocks sit; no computation. □

## 5. The exclusion

### 5.1. The twenty-four refutations.

**Proposition 5.1.** *For each of the twenty-four quadruples $(n, a, w, t)$ with $(n, a, w)$ in (2) and $t \in \{0, 1\}$, the formula $\Phi(n, a, w, t)$ is unsatisfiable, and hence* $\mathrm{Cell}(n, a, w, t) = \emptyset$.

*Proof sketch.* This is the one place where *exhaustive computation* carries the argument, and it is worth saying exactly what was run. For each of the twenty-four quadruples, the DIMACS file was emitted directly from the definition of $\Phi$, so the formula the solver was given and the formula the fidelity argument speaks about are the same object. CaDiCaL 2.1.2 was run on that file with LRAT proof logging, sixteen random seeds per quadruple and the smallest proof kept; every run returned unsatisfiable. The resulting twenty-four LRAT proofs, listed in Table 1, total 59.0 MB, the largest being 8.27 MB, and each is replayed by an LRAT checker that is itself verified in the proof assistant, so no property of CaDiCaL is assumed. Total solving time was under one CPU-minute for the best single seed and about thirteen CPU-minutes for the sixteen-seed sweep. Emptiness of the cell then follows from Proposition 4.3. □

Table 1. The twenty-four cells: variables, clauses of $\Phi$, and the size of the stored LRAT refutation.

| $(n, a, w, t)$ | vars | clauses | LRAT | $(n, a, w, t)$ | vars | clauses | LRAT |
|---|---|---|---|---|---|---|---|
| $(7, 3, 2, 0)$ | 21 | 947 | 14 kB | $(7, 3, 2, 1)$ | 21 | 929 | 20 kB |
| $(8, 3, 3, 0)$ | 28 | 3 574 | 85 kB | $(8, 3, 3, 1)$ | 28 | 3 544 | 106 kB |
| $(8, 4, 2, 0)$ | 28 | 3 527 | 46 kB | $(8, 4, 2, 1)$ | 28 | 3 509 | 62 kB |
| $(9, 4, 3, 0)$ | 36 | 10 440 | 0.22 MB | $(9, 4, 3, 1)$ | 36 | 10 410 | 0.41 MB |
| $(10, 4, 4, 0)$ | 45 | 25 914 | 0.57 MB | $(10, 4, 4, 1)$ | 45 | 25 856 | 0.95 MB |
| $(10, 5, 3, 0)$ | 45 | 25 763 | 0.51 MB | $(10, 5, 3, 1)$ | 45 | 25 733 | 0.99 MB |
| $(11, 5, 4, 0)$ | 55 | 56 641 | 1.28 MB | $(11, 5, 4, 1)$ | 55 | 56 583 | 1.79 MB |
| $(12, 5, 5, 0)$ | 66 | 113 268 | 2.73 MB | $(12, 5, 5, 1)$ | 66 | 113 150 | 3.07 MB |
| $(12, 6, 4, 0)$ | 66 | 112 809 | 2.56 MB | $(12, 6, 4, 1)$ | 66 | 112 751 | 3.22 MB |
| $(13, 6, 5, 0)$ | 78 | 210 024 | 4.52 MB | $(13, 6, 5, 1)$ | 78 | 209 906 | 4.85 MB |
| $(14, 6, 6, 0)$ | 91 | 368 544 | 8.24 MB | $(14, 6, 6, 1)$ | 91 | 368 302 | 7.97 MB |
| $(14, 7, 5, 0)$ | 91 | 367 171 | 6.52 MB | $(14, 7, 5, 1)$ | 91 | 367 053 | 8.27 MB |

### 5.2. The parameter statement.

**Theorem 5.2.** *Let $(n, a, w)$ be one of the twelve triples of (2) and let $t \in \{0, 1\}$. Then* $\mathrm{Cell}(n, a, w, t)$ *is empty. Consequently, for each of those twelve triples, no $\mathcal{B}$-free graph $G$ on $n$ vertices satisfies $\alpha(G) = a$ and $\omega(G) = w$.*

*Proof.* The first statement is Proposition 5.1, quadruple by quadruple. The second follows from it by Proposition 4.4, whose hypotheses are exactly $\alpha(G) = a$ and $\omega(G) = w$ split into an extremal set and a bound, together with Lemma 2.2. □

Theorem 5.2 is stronger than the finite step [2] needs, in two independent ways.

*Remark* 5.3. *No corona constraints.* By Remark 2.4 the systems refuted here are strictly weaker than those of [2, Prop. 4.1], so Theorem 5.2 implies the infeasibility the source needs but is not implied by it. In particular, with the exclusion in the form proved here, the whole of [2, §2] is needed only to bound the size of a counterexample and enters the finite exclusion not at all — whereas in [2] its Proposition 2.3 also supplies the corona block of the systems that its Proposition 4.1 refutes.

*Both values of $t$.* The source's table forms only twenty-one of the twenty-four quadruples. Its Lemma 3.2 shows that a smallest counterexample with $|S \cap K| = 1$ satisfies $n \le 12$: from the two sets [2, Prop. 2.3] supplies inside $K$ and inside $S$ it counts the edges between them in two ways, and the resulting inequality bounds $\alpha(G)$. So the quadruples $(13, 6, 5, 1)$, $(14, 6, 6, 1)$ and $(14, 7, 5, 1)$ are never formed there. All three are refuted here as well, which is what makes the emptiness statement of Theorem 5.2 unconditional in $t$ and lets Proposition 4.4 be applied without any further argument.

*Remark* 5.4 (measured, not assumed). Dropping the corona constraints was not a guess. The systems of [2, §4] were first re-encoded from their printed form and solved — all twenty-one are unsatisfiable, 126.9 CPU-seconds with CaDiCaL 2.1.2 and no symmetry breaking — and only then were constraint families removed one at a time and the systems re-solved with a 120-second limit. The outcome on four representative quadruples:

| cell | drop $\mathcal{B}$-freeness | drop corona | drop $\alpha \le a$ | drop $\omega \le w$ | drop nothing |
|---|---|---|---|---|---|
| $(7, 3, 2, 0)$ | SAT | UNSAT | SAT | SAT | UNSAT |
| $(8, 4, 2, 0)$ | SAT | UNSAT | UNSAT | SAT | UNSAT |
| $(10, 4, 4, 0)$ | SAT | UNSAT | SAT | SAT | UNSAT |
| $(11, 5, 4, 0)$ | SAT | UNSAT | UNSAT | SAT | UNSAT |

The corona column is the finding of this paper; every other column says the remaining hypotheses are load-bearing, and Proposition 7.1(iii) turns that observation into explicit graphs. These solver runs are exploratory measurements and are *not* part of the formal development: nothing in Theorem 5.2 depends on them.

*Remark* 5.5 (symmetry breaking). Without the two lexicographic families the refutations are of no practical use. A cell of type $(n, a, w, t)$ has the automorphism group $\mathrm{Sym}(S \setminus K) \times \mathrm{Sym}(K \setminus S) \times \mathrm{Sym}(R)$ of order $(a - t)!\,(w - t)!\,(t + 2)!$, up to $7! \cdot 5! \cdot 2! = 1\,209\,600$ at $(14, 7, 5, 0)$. Measured with CaDiCaL 2.1.2: with no break at all, the fourteen quadruples that finished inside a ten-minute window already produced 1.66 GB of LRAT, the single proof at $(14, 6, 6, 0)$ accounting for 1.14 GB; with the $S \setminus K$ break alone, all twenty-four finish and produce 209.5 MB; with both breaks, 63.9 MB for a single seed and 59.0 MB for the best of sixteen. The nesting of Proposition 4.3 — ordering $K \setminus S$ by a key that the second permutation cannot disturb — is what allows both breaks at once.

## 6. The conjecture, conditionally

Theorem 5.2 is the finite half of [2]. The other half is its structural reduction, which we neither prove nor use in any other form than the following named hypothesis.

**Theorem 6.1.** *Suppose the following reduction hypothesis holds.*

> (R) *For every $n, a, w$ and every graph $G$ on $[n]$ that is $\mathcal{B}$-free with $\alpha(G) = a$, $\omega(G) = w$ and $a + w + 2 \le n$, there exist $n', a', w'$ with $(n', a', w')$ in (2) and a $\mathcal{B}$-free graph $G'$ on $[n']$ with $\alpha(G') = a'$ and $\omega(G') = w'$.*

*Then Conjecture 1.1 holds: every $\mathcal{B}$-free graph $G$ satisfies $\alpha(G) + \omega(G) \ge |V(G)| - 1$.*

*Proof.* Suppose $G$ is $\mathcal{B}$-free on $[n]$ with $\alpha(G) = a$, $\omega(G) = w$ and $n > a + w + 1$, i.e. $a + w + 2 \le n$. By (R) there is a $\mathcal{B}$-free $G'$ on $[n']$ with $\alpha(G') = a'$, $\omega(G') = w'$ and $(n', a', w')$ in (2), contradicting Theorem 5.2. □

Hypothesis (R) is exactly what Sections 2 and 3 of [2] establish, and its ingredients are a Hall-type exchange corollary, the set-and-collection lemma of Remark 2.4 and two double counts. We

state it plainly: *it is assumed, not proved, in this paper.* The obstruction to removing the assumption is not the exchange argument or the double counts, both of which are routine to formalise given Hall's marriage theorem; it is the set-and-collection lemma [3]. Mathlib at the revision this development builds against (`81a5d257`, for Lean `v4.32.0`) has Hall's marriage theorem and the parameters `SimpleGraph.indepNum` and `SimpleGraph.cliqueNum`, but — checked on 29 August 2026 — carries no notion of the core or the corona of the family of maximum stable sets and no form of the set-and-collection lemma, so it would have to be developed from scratch. We have not done so, and Theorem 6.1 is therefore a conditional statement in the strict sense, with the condition machine-readable rather than rhetorical.

## 7. Sharpness and controls

The following are the checks a reader should be able to perform without trusting any certificate. They are not new; they are recorded because each of them fails for a nearby statement, and their failure is what tells the reader that Theorem 5.2 is not an artefact of an over-strong hypothesis.

**Proposition 7.1.** (i) *$C_5$ is $\mathcal{B}$-free with $\alpha = \omega = 2$ and $n = 5$, so $\alpha + \omega = n - 1$. Hence the conclusion of Conjecture 1.1 cannot be strengthened to $\alpha + \omega \geq n$; that is, $\mathcal{B}$-free graphs need not be sum-perfect.*

(ii) *$C_7$ has $\alpha = 3$, $\omega = 2$ and $n = 7$, so $\alpha + \omega = n - 2$, one below the bound; and $C_7$ is not $\mathcal{B}$-free. Hence the $\mathcal{B}$-freeness hypothesis of Conjecture 1.1 cannot be dropped.*

(iii) *Each of the three conditions* (C3), (C4), (C5) *of Definition 2.3 is load-bearing at $(n, a, w, t) = (7, 3, 2, 0)$. With $S = \{0, 1, 2\}$ stable and $K = \{3, 4\}$ a clique on the vertex set* $[7]$*, the graphs with edge sets*

$$\{34\}, \qquad \{06, 13, 14, 15, 34, 35, 45\}, \qquad \{04, 06, 13, 16, 25, 34\}$$

*satisfy all of* (C1)–(C5) *except, respectively,* (C3) *(the first has $\alpha = 6$),* (C4) *(the second has $\omega = 4$) and* (C5) *(the third is not $\mathcal{B}$-free).*

*Proof sketch.* (i) $C_5$ has fewer than six vertices, hence is $\mathcal{B}$-free by Proposition 3.3; its $\alpha$ and $\omega$ are decided by inspection of the $2^5$ subsets of its vertex set. (ii) $\alpha$ and $\omega$ of $C_7$ likewise, by inspection of the $2^7$ subsets; and $C_7$ is not $\mathcal{B}$-free because deleting one vertex leaves $P_6$, exhibited by the stable triples $\{0, 2, 4\}$ and $\{1, 3, 5\}$ with the matching 0–1, 2–3, 4–5. (iii) For each of the three graphs, the four assertions "$\{0, 1, 2\}$ stable", "$\{3, 4\}$ a clique", and the two bounds are decided by *exhaustive computation* over the $2^7 = 128$ subsets of $[7]$, and $\mathcal{B}$-freeness (or its failure) by the complete $7^6 = 117\,649$-tuple search of Proposition 3.3. The three graphs themselves were found by a search over all $2^{21} = 2\,097\,152$ graphs on seven vertices; that search is not part of the formal development — only the three resulting graphs, and the assertions above about them, are verified. The same search found no graph on seven vertices satisfying all of (C1)–(C5), which recomputes $\mathrm{Cell}(7, 3, 2, 0) = \emptyset$ by a completely different method. □

*Remark* 7.2. Part (i) explains the shape of Conjecture 1.1: $C_5$ is $H_1$ in the list $\mathcal{F}$ of [1], and it is one of the three graphs removed from $\mathcal{F}$ to form $\mathcal{B}$, so it must be a $\mathcal{B}$-free graph witnessing the $-1$.

## 8. Verification

Every statement of this paper has been formally verified in Lean 4 [6] (toolchain `v4.32.0`) against *Mathlib* [7] at revision `81a5d257`; the development contains no `sorry` and declares no axiom of its own. Repository reference to be added at submission.

The reasoning layer is kernel-clean: Lemma 2.2, the fidelity argument (Proposition 4.2), the symmetry-breaking licence (Proposition 4.3), the removal of the enumeration (Proposition 4.4), the bridge between the Boolean pattern test and Definition 2.1 used in Theorem 3.1, and the two controls of Proposition 7.1(i)–(ii) depend on nothing beyond `propext`, `Classical.choice` and `Quot.sound`; the non-$\mathcal{B}$-freeness of $C_7$ depends on no axiom at all. What is delegated to compiled evaluation —

Lean's `native_decide`, which emits one auxiliary axiom per evaluated Boolean — is exactly the finite searches: one Boolean per quadruple in Proposition 5.1, namely that the verified LRAT checker accepts the stored certificate; one for the sweep of Theorem 3.1; one for the tally of Theorem 3.2; and the $\mathcal{B}$-pattern searches of the three witnesses of Proposition 7.1(iii), whose assertions about stable sets and cliques are decided in the kernel. The parameter statement of Theorem 5.2 accumulates exactly the twenty-four certificate axioms above the three standard ones, and nothing else.

Two independent cross-checks were run on the encoding, which is the point where a silent error would be most damaging. First, the DIMACS file handed to CaDiCaL is emitted from the same definition of $\Phi$ that the fidelity argument speaks about and that the checker replays, so there is no second generator to keep in step; clause order is load-bearing, because LRAT names input clauses by position, and a mismatch makes the checker reject rather than produce a wrong theorem. Second, an independent encoder was written in Python directly from the printed constraints of [2, §4], without reading or running the source's ancillary code. Canonicalised — the literals of each clause sorted as integers, the clauses then sorted as integer tuples, one clause to a line and no DIMACS header — the two encoders agree byte for byte; the SHA-256 digests (first 16 hexadecimal digits) of the canonical forms are

| cell | clauses | SHA-256 prefix, both encoders |
|---|---|---|
| $(7,3,2,0)$ | 947 | `6564c492a45649a2` |
| $(11,5,4,1)$ | 56 583 | `e06021929ef25e07` |
| $(14,6,6,0)$ | 368 544 | `748abf1f6bc4be28` |
| $(14,7,5,1)$ | 367 053 | `472865f85c505787` |

(The two differ in clause order and in literal order inside the complementary clauses; as multisets of clauses they are identical. The four digests were recomputed on 29 August 2026 from a fresh dump of $\Phi$.) The same Python encoder, applied to the source's own systems, reproduces its variable counts cell for cell, and its clause counts up to exactly the symmetry-breaking constraints the source describes in the proof of its Proposition 4.1, where it "orders the $y$-selectors on $K \setminus S$ and on two exceptional labels, and the $z$-selectors on $S \setminus K$" — the corona selectors of its (4.5) and (4.8). That is $(w-t-1)+(a-t-1)+1 = a+w-2t-1$ clauses per cell, from 2 at $(7,3,2,1)$ to 11 at $(14,6,6,0)$ and $(14,7,5,0)$. Two independent readings of the same printed system produce the same system. Finally, the count 4400 of Theorem 3.2 and the encoding agreement of Theorem 3.1 were first obtained in Python and only then proved.

## Acknowledgement of priority

Conjecture 1.1 is due to Litjens, Polak and Sivaraman [1]. The reduction to twelve parameter triples, the table (2) itself, and the assertion that the corresponding systems are infeasible are due to Frijio [2], whose preprint appeared in August 2026; twenty-one of the twenty-four quadruples of Proposition 5.1 are covered, in a more constrained form, by its Proposition 4.1, and the encoding agreement of Theorem 3.1 and the total 4400 of Theorem 3.2 are reported there. It should be said plainly that if the finite step of [2] is granted, then its Theorem 1.1 — Conjecture 1.1 itself — implies the second assertion of Theorem 5.2 at once, since $\alpha+\omega = a+w = n-2$ there. What this paper contributes is the corona-free and $t$-unconditional form of the exclusion (Theorem 5.2), the demonstration that each surviving hypothesis is load-bearing (Remark 5.4, Proposition 7.1(iii)), the refinement $4400 = 2200 + 2200$ (Theorem 3.2), and checkable evidence for all of it.

As of 29 August 2026, [2] stands at v1, the version posted on 23 August 2026; the e-print carries no journal reference and no publisher DOI, and no citation to it is recorded by either Semantic Scholar or OpenAlex.

## References

[1] B. Litjens, S. Polak and V. Sivaraman, *Sum-perfect graphs*, Discrete Applied Mathematics **259** (2019) 232–239, doi:10.1016/j.dam.2018.12.015; e-print arXiv:1710.07546.

[2] D. Frijio, *A Proof of the B-Free Graphs Conjecture*, e-print arXiv:2608.22537v1 (23 August 2026).

[3] V. E. Levit and E. Mandrescu, *A set and collection lemma*, The Electronic Journal of Combinatorics **21** (2014), no. 1, Paper #P1.40, doi:10.37236/2514; e-print arXiv:1101.4564.
[4] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleyks and F. Pollitt, *CaDiCaL 2.0*, in: Computer Aided Verification (CAV 2024), Lecture Notes in Computer Science **14681**, Springer, 2024, 133–152, doi:10.1007/978-3-031-65627-9_7.
[5] L. Cruz-Filipe, M. J. H. Heule, W. A. Hunt Jr., M. Kaufmann and P. Schneider-Kamp, *Efficient certified RAT verification*, in: Automated Deduction — CADE 26, Lecture Notes in Computer Science **10395**, Springer, 2017, 220–236, doi:10.1007/978-3-319-63046-5_14.
[6] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.
[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.