HOMOLOGY PROFILES AND GRADED BETTI TABLES OF SQUAREFREE MONOMIAL IDEALS IN FIVE AND SIX VARIABLES

KOREA SUPERINTELLIGENCE LABS

Abstract. Ripke and Yoon recently determined that exactly 134 distinct graded Betti tables occur among the 208 nonzero proper squarefree monomial ideals in five variables up to relabelling of the variables, and closed their paper with several questions, two of which we answer here: whether those 134 types are classified by the homology profiles of induced subcomplexes, and how many distinct graded Betti tables occur for general n. The homology profile — the multiset, for each cardinality j, of the reduced homology dimension vectors of the induced subcomplexes on j vertices — is a strictly finer invariant than the graded Betti table and a strictly coarser one than the relabelling class: in five variables the 208 classes realize exactly 189 profiles and 134 tables, so of the 74 coincidences of Betti tables the loss of positional information named by Ripke and Yoon accounts for 19 and the summation for the remaining 55. Exactly 97 of the 134 table types come from a single profile. In six variables, the first open cell of the enumerative question, the 16 351 nonzero proper classes realize exactly 8233 homology profiles and exactly 1604 graded Betti tables, over every field; the enumeration is certified by orbit counting against the Dedekind number $M(6) = 7\,828\,354$, itself computed rather than quoted. Every count stated here is machine-checked in Lean 4, with Hochster's formula quoted from the literature and one further step, named where it is used, argued rather than formalized.

1. Introduction

Let k be a field, $S = k[x_1, \dots, x_n]$, and let $I \subseteq S$ be a squarefree monomial ideal. The graded Betti numbers $\beta_{i,j}(S/I)$ are the ranks of the free modules in the minimal graded free resolution of S/I, arranged in the *graded Betti table* of S/I. Relabelling the variables carries I to an ideal with an isomorphic quotient and therefore leaves the graded Betti table unchanged, so the table is an invariant of the S_n-orbit of I.

Ripke and Yoon [5, Thm. 1.1] proved that for $T = k[x_1, \dots, x_5]$ and a monomial ideal $M \subseteq T$ the multigraded, graded and total Betti numbers of T/M are independent of the characteristic of k; six is therefore, in their words, "the smallest number of variables in which characteristic dependence can occur". The bound is sharp at six: Peeva's example, which [5, §7] reproduces from [4, Ex. 12.4], has graded Betti numbers that differ in characteristic 2, the obstruction being the $\mathbb{Z}/2$-torsion in the homology of the minimal six-vertex triangulation of $\mathbb{RP}^2$. Along the way they enumerated the squarefree case completely: up to relabelling there are 208 nonzero proper squarefree monomial ideals in $k[x_1, \dots, x_5]$, and among the quotients S/I exactly 134 distinct graded Betti tables occur [5, Prop. 8.1]. Their concluding section poses three questions. The first [5, §10] asks

> "can the 134 Betti table types be classified by the homology profiles of induced subcomplexes on five vertices? Hochster's formula shows that a graded Betti table records only the total reduced homology dimensions of induced subcomplexes of each cardinality, not which induced subcomplexes contribute. This loss of information explains why nonisomorphic Stanley–Reisner complexes can have the same graded Betti table."

The second asks which structural families account for repeated Betti tables. The third and last [5, §10] asks

> "Finally, for general n, how many distinct graded Betti tables occur among squarefree monomial ideals in n variables up to relabeling? The case $n = 5$ gives 134 table types among 208 nonzero proper relabeling classes, suggesting a broader enumerative problem."

This paper answers the first question, and the third at $n = 6$, its first open cell.

The invariant, and what the first question is asking. The quoted sentence names two distinct losses of information and offers the second as the explanation of the first. Separating them is the content of our first result. For a simplicial complex Δ on $[n] = \{1, \dots, n\}$, an induced subcomplex $\Delta|_\sigma$ and a field k, write

$$h_\Delta(\sigma) \;=\; \bigl(\dim_k \widetilde{H}_{r-1}(\Delta|_\sigma; k)\bigr)_{r=0,\dots,n} \qquad (\sigma \subseteq [n]),$$

a vector recording the whole reduced homology of one induced subcomplex. Three invariants sit in a chain: the *labelled profile* $\mathcal{L}(\Delta)$, the function $\sigma \mapsto h_\Delta(\sigma)$ taken up to the S_n-action; the *homology profile* $\mathcal{P}(\Delta)$, which for each j records only the multiset $\{h_\Delta(\sigma) : |\sigma| = j\}$; and the graded Betti table $\mathcal{B}(\Delta)$, which for each j records only the entrywise sum of that multiset. By Hochster's formula the last of these is the graded Betti table of S/I when Δ is the Stanley–Reisner complex of I (Section 2), so

$$\mathcal{L} \;\twoheadrightarrow\; \mathcal{P} \;\twoheadrightarrow\; \mathcal{B},$$

and the two arrows are exactly the two losses Ripke and Yoon name: the first forgets *which* induced subcomplexes contribute, the second is the passage to the *total*. Their first question is whether the second arrow is injective — whether the graded Betti table already determines the homology profile.

Results. Our first result answers that, negatively and quantitatively.

Theorem 1.1 (Theorem 3.1)**.** *The* 208 *nonzero proper relabelling classes of squarefree monomial ideals in five variables realize exactly* 189 *distinct homology profiles and exactly* 134 *distinct graded Betti tables. Of the* $208 - 134 = 74$ *coincidences of Betti tables,* $208 - 189 = 19$ *already occur at the level of homology profiles — distinct classes with the same profile, the loss Ripke and Yoon name — and the remaining* 55 *appear only after the summation. Exactly* 97 *of the* 134 *table types come from a single homology profile,* 22 *from two,* 12 *from three and* 3 *from four.*

So the classification the question asks for exists for 97 of the 134 types and fails for the other 37, and the explanation offered in [5, §10] covers about a quarter of the phenomenon it is offered for. Section 3 exhibits both failures by named ideals: four pairwise distinct relabelling classes sharing one homology profile, and the unique six-fold Betti table of [5, §8] splitting into exactly three profiles, $6 = 3 + 2 + 1$. At $n = 4$ the homology profile is still a complete invariant of the relabelling class, so five variables is where the first arrow starts to lose.

Our second result settles the enumerative question at its first open cell.

Theorem 1.2 (Theorem 4.3)**.** *Up to relabelling there are* 16 353 *squarefree monomial ideals in six variables,* 16 351 *of them nonzero and proper. Over any field, these realize exactly* 8233 *distinct homology profiles and exactly* 1604 *distinct graded Betti tables.*

The chain splits $16\,351 \twoheadrightarrow 8233 \twoheadrightarrow 1604$; at $n = 6$ the positional loss (8118 classes) is the larger of the two, where at $n = 5$ the summation dominated 55 to 19. The passage from one field to another is not automatic here, because six variables is exactly where characteristic dependence begins — not below six by [5, Thm. 1.1], and at six by Peeva's example [5, §7]: the Stanley–Reisner ideal of the minimal six-vertex triangulation of $\mathbb{R}\mathrm{P}^2$ has a graded Betti table that differs in characteristic 2. Section 5 shows that it is the only one of the 16 351 classes that does, and that the exchange it forces leaves both counts unchanged — for the Betti tables because the *set* of tables is unchanged, and for the profiles by a cancellation, one profile leaving the set and another entering it.

Underneath Theorem 4.3 is an enumeration that has to be certified rather than trusted, since 7 828 354 ideals cannot be canonically reduced inside a proof assistant by brute force. Section 4 supplies 16 353 orbit representatives as data and proves them complete and irredundant by orbit counting against the Dedekind number $M(6) = 7\,828\,354$, which is itself computed in the formal development rather than imported from the literature. Nothing about how the list of representatives was produced enters the argument.

What is proved and what is taken as given. Hochster's formula is used as the definition of the graded Betti table; it is a theorem of the literature [2, Cor. 5.12], quoted as [5, Thm. 4.4], and is not reproved here. What the formal development does prove about it, in general and for all n, is that the object whose homology is taken really is a simplicial complex, that the empty face belongs to it exactly when the ideal is proper, and that the Euler–Poincaré identity relating the computed homology dimensions to the face numbers holds; and what it checks on all 208 five-variable classes is that row 1 of the resulting table counts the minimal generators by degree, a fact about a free resolution that no homology computation is told (Section 6). We also do not claim priority for a search: the counts 189, 8233 and 1604 were not found in the literature we consulted, and [6], the data repository accompanying [5], contains five-variable material only, but "not found" is not "not there".

Everything asserted below as a theorem, proposition, lemma or corollary is machine-checked in Lean 4 (Section 8), modulo one identification of two boundary matrices — the step that carries the passage from one field to every field — which is argued rather than formalized and is stated as Remark 5.3 where it is first used. Remark 4.4 reports a computation made outside the formal development and is used in no proof.

2. Preliminaries

2.1. **Ideals, antichains, complexes.** Fix n and write $x_F = \prod_{i\in F} x_i$ for $F \subseteq [n]$. A squarefree monomial ideal $I \subseteq S = k[x_1,\dots,x_n]$ is determined by its minimal generating set, and since divisibility of squarefree monomials is inclusion of their supports, that generating set is an antichain $\mathcal{A} \subseteq 2^{[n]}$: $I = (x_F : F \in \mathcal{A})$. This is a bijection between squarefree monomial ideals and antichains of $2^{[n]}$, so the number of squarefree monomial ideals in n variables is the Dedekind number $M(n)$ [3, A000372]. The zero ideal is $\mathcal{A} = \emptyset$ and the unit ideal is $\mathcal{A} = \{\emptyset\}$; all other $\mathcal{A}$ give ideals that are *nonzero and proper*, and those are the ones counted below.

The *Stanley–Reisner complex* of I is

$$\Delta_I \;=\; \{\, \sigma \subseteq [n] \;:\; F \not\subseteq \sigma \text{ for every } F \in \mathcal{A} \,\}, \qquad \Delta_I|_\sigma \;=\; \{\, \tau \in \Delta_I : \tau \subseteq \sigma \,\}.$$

The symmetric group S_n acts on antichains by permuting $[n]$; ideals in one orbit have isomorphic quotients. Following [5] we count *relabelling classes*, that is S_n-orbits. The number of orbits of antichains of $2^{[n]}$ is [3, A003182], the sequence [5] cites for its own count 210 at $n = 5$.

2.2. **Hochster's formula and the convention at degree -1.** Hochster's formula [2, Cor. 5.12], in the form quoted as [5, Thm. 4.4], reads

$$\beta_{i,x_\sigma}(S/I) \;=\; \dim_k \widetilde{H}_{|\sigma|-i-1}(\Delta_I|_\sigma; k), \qquad \beta_{i,j}(S/I) \;=\; \sum_{|\sigma|=j} \beta_{i,x_\sigma}(S/I).$$

(Miller and Sturmfels state it in reduced *cohomology*, $\dim_k \widetilde{H}^{|\sigma|-i-1}$; over a field the two dimensions agree, and [5, Thm. 4.4] is the homological form written above.) It requires the convention $\widetilde{H}_{-1}(\Gamma; k) = k$ when $\Gamma = \{\emptyset\}$ and 0 otherwise, which [5, §4] states explicitly and remarks is "needed in Hochster's formula when an induced subcomplex has no nonempty faces". The convention is not decorative: dropping it changes the answer at $n = 5$ from 134 to 130, and dropping the empty face from the complex instead — which silently unreduces $\widetilde{H}_0$ — returns the published count 134 with every one of the 208 tables wrong. Both are recorded in Section 7, because the second means that reproducing the count 134 is by itself no evidence that an implementation is correct.

2.3. **The three invariants.** With $h_\Delta(\sigma)$ as in Section 1, define for a Stanley–Reisner complex Δ on $[n]$:

$$\begin{aligned}
\mathcal{L}(\Delta) &= \text{the function } \sigma \mapsto h_\Delta(\sigma), \text{ up to the } S_n\text{-action;}\\
\mathcal{P}(\Delta) &= \text{the multiset } \{\, (|\sigma|,\, h_\Delta(\sigma)) : \sigma \subseteq [n] \,\};\\
\mathcal{B}(\Delta) &= \Big(\textstyle\sum_{|\sigma|=j} h_\Delta(\sigma)\Big)_{j=0,\dots,n}, \quad \text{the graded Betti table, by Hochster's formula.}
\end{aligned}$$

We call $\mathcal{P}(\Delta)$ the *homology profile*, in the sense of the question of [5, §10]. Each of the three is computed from the same reduced homology data, so $\mathcal{L} \twoheadrightarrow \mathcal{P} \twoheadrightarrow \mathcal{B}$ by construction, and the numbers of distinct values are weakly decreasing along the chain. Two remarks make the chain worth stating. First, $\mathcal{P}$ is genuinely an invariant of the relabelling class and not merely of a chosen representative: this is checked exhaustively at $n \leq 5$, every class against every one of the $n!$ relabellings (Section 7). Second, summing $\mathcal{P}$ really does return the graded Betti table of the ideal — checked on all 208 classes at $n = 5$ and on 4000 representatives at $n = 6$ — so the three counts are three readings of one computation and not three computations.

2.4. **Homology dimensions and the base field.** All reduced homology below is simplicial homology of a finite complex, so $\dim_k \widetilde{H}_r(\Gamma; k)$ is determined by the ranks over k of the integer boundary matrices, and depends on k only through its characteristic. Section 5 makes the passage to an arbitrary field explicit by rank certificates; until then, statements are over a fixed but arbitrary k and the results of Section 5 are what justifies dropping k from the notation.

3. Five variables: the homology profile against the Betti table

Theorem 3.1. *Among the* 208 *nonzero proper relabelling classes of squarefree monomial ideals in five variables:*

(i) *exactly* 189 *distinct homology profiles and exactly* 134 *distinct graded Betti tables occur, so the homology profile is strictly finer than the graded Betti table and strictly coarser than the relabelling class;*
(ii) *of the* $74 = 208 - 134$ *coincidences, exactly* 19 *are already coincidences of homology profiles and exactly* 55 *are collapses of distinct profiles onto one table;*
(iii) 97 *of the* 134 *graded Betti tables are realized by a single homology profile,* 22 *by two,* 12 *by three and* 3 *by four;*
(iv) 175 *of the* 189 *homology profiles are realized by a single relabelling class,* 10 *by two,* 3 *by three and* 1 *by four;*
(v) *in four variables the homology profile is a complete invariant of the relabelling class,* 28 *profiles for* 28 *classes, while in five variables it is not.*

Proof sketch. The proof is an exhaustive computation, and the finite search it runs is this. The 7581 antichains of $2^{[5]}$ are enumerated by a pruned depth-first search over the subsets of [5], each is reduced to the numerically least image of its bitmask under the 120 relabellings, and the 210 canonical representatives so obtained are reduced to the 208 nonzero proper ones. For each of those, the reduced homology dimensions of all $2^5 = 32$ induced subcomplexes are computed — 6656 homology computations in all — and the three invariants of Section 2 are read off the same data. Counting distinct values gives 208, 189 and 134; the fibre statistics (iii) and (iv) are read off the same pass, and (v) is the same computation at $n = 4$. All of (i)–(v) are projections of a single compiled evaluation, so they rest on one exhaustive search and not on five. □

Both arrows of the chain are witnessed by named ideals, so neither (i) nor (ii) rests on a count alone. Write abc for $x_1x_2x_3$ and use a, b, c, d, e for the five variables.

Proposition 3.2. *The four ideals*

$$(bc,\ bd,\ acd,\ abe), \quad (abc,\ abd,\ ae,\ be), \quad (abc,\ acd,\ ae,\ be), \quad (acd,\ bcd,\ ae,\ be)$$

lie in four pairwise distinct nonzero proper relabelling classes and have one and the same homology profile. Conversely, the six ideals

$$\begin{aligned} &(abc, abd, acd, abe, ce), \quad (abc, acd, bcd, abe, ce), \quad (bc, abd, acd, abe, ace), \\ &(abd, acd, bcd, abe, ce), \quad (abd, bcd, abe, ace, de), \quad (abc, abd, acd, bcd, ae) \end{aligned}$$

lie in six pairwise distinct nonzero proper classes sharing one graded Betti table — which by Proposition 6.1 can only be the unique table of [5, §8] *realized by six classes — and carry exactly three homology profiles, the profile map splitting them into blocks of sizes* 3*,* 2 *and* 1*. So three of the five coincidences* *inside that six-fold table are the positional loss named in* [5, §10] *and the other two are coincidences of sums.*

In the order the six are listed, the blocks are the first three, the next two and the last; that assignment is read off the same evaluation and, unlike the block sizes, is not separately pinned by a theorem.

The first half of Proposition 3.2 refutes any count above 189 in Theorem 3.1(i) without appealing to a total: four classes, one profile. The second half does the same for the other arrow — one Betti table, three profiles — so that neither strict inequality of Theorem 3.1(i) rests on a count alone.

The chain of Theorem 3.1 begins at 208 rather than lower because the labelled profile loses nothing at all in this range.

Proposition 3.3. *For $n = 3, 4, 5$ the labelled profile $\mathcal{L}$ is a complete invariant of the relabelling class of a nonzero proper squarefree monomial ideal in n variables: there are* 8, 28 *and* 208 *distinct labelled profiles for* 8, 28 *and* 208 *classes. That is, recording for every $\sigma \subseteq [n]$ only the reduced homology* dimensions *of $\Delta_I|_\sigma$, and not its faces, already determines I up to relabelling.*

Proof sketch. Exhaustive: the same pass as Theorem 3.1 compares labelled profiles across all classes at $n = 3, 4, 5$. We record the argument that makes the general statement plausible, and emphasise that it is *not* what is verified here. Let σ have cardinality c and suppose every proper subset of σ is a face of Δ. Then $\Delta|_\sigma$ is either the full simplex on σ, which is contractible, or its boundary sphere S^{c-2}, so $h_\Delta(\sigma)$ decides which; and if some proper subset of σ is not a face then σ is not a face either, by down-closedness. Induction on $|\sigma|$ then recovers Δ from $\mathcal{L}(\Delta)$, with the base case $\emptyset \in \Delta$ read off $h_\Delta(\emptyset)$. Formalizing this needs the reduced homology of the boundary of a simplex for general c, which the present development computes rather than proves; what is verified is the finite check at $n \leq 5$. The same count holds at $n = 6$ — 16 351 labelled profiles for 16 351 classes — by the independent program of Remark 4.4, outside the formal development. □

4. Six variables: the enumerative question at its first open cell

At $n = 6$ the enumeration is the obstruction rather than the homology: there are $M(6) = 7\,828\,354$ antichains of $2^{[6]}$, and canonically reducing each of them under all 720 relabellings inside a proof assistant is out of the question. We therefore supply a list of 16 353 candidate orbit representatives as data and *prove* it complete and irredundant, so that no property of the program that produced it is used.

Lemma 4.1. *$M(3) = 20$, $M(4) = 168$, $M(5) = 7581$ and $M(6) = 7\,828\,354$, where $M(n)$ is the number of antichains of $2^{[n]}$.*

These are the Dedekind numbers [3, A000372]; [5, Prop. 8.1] quotes that entry for $M(5) = 7581$. What matters here is not the values but that they are recomputed: the counting enumeration used for Lemma 4.1 replaces the list of antichains by its length and so never materializes the 7.8 million objects, and at $n = 5$ it is checked to agree with the list-producing enumeration of Section 6. Consequently the completeness argument below imports no classical data at all.

Proposition 4.2. *There is an explicit list of* 16 353 *antichains of $2^{[6]}$ that meets every S_6-orbit exactly once. Hence there are exactly* 16 353 *relabelling classes of squarefree monomial ideals in six variables, and* 16 351 *nonzero proper ones.*

Proof sketch. Four exhaustive checks on the supplied list. (a) Every entry is an antichain, and the 16 353 entries are strictly increasing as integers, hence pairwise distinct. (b) Every entry is the numerically least element of its S_6-orbit; this is $16\,353 \times 720$ relabellings, run in eight blocks of about 2045 entries. (c) The orbit sizes of the entries sum to $750907 + 1031049 + 1047450 + 1099950 + 923233 + 1020861 + 1109232 + 845672 = 7\,828\,354$, one summand per block. (d) That total is $M(6)$, by Lemma 4.1. Distinct canonical forms cannot lie in one orbit, so (a) and (b) give 16 353 pairwise distinct orbits; (c) says those orbits contain 7 828 354 antichains between them; (d) says that is all of them. A wrong list fails one of (a), (b), (c). □

Theorem 4.3. *Let k be a field. Among the* 16 351 *nonzero proper relabelling classes of squarefree monomial ideals in $k[x_1, \dots, x_6]$, exactly* 8233 *distinct homology profiles and exactly* 1604 *distinct graded Betti tables occur. Moreover:*

(i) *the chain splits* $16\,351 - 8233 = 8118$ *and* $8233 - 1604 = 6629$*, of the* 14 747 *coincidences in all;*
(ii) 5420 *of the homology profiles and* 548 *of the graded Betti tables are realized by a single relabelling class; the largest fibre of the profile map has* 28 *classes and the largest fibre of the Betti table map has* 231*;*
(iii) 575 *of the* 1604 *graded Betti tables are realized by a single homology profile.*

Proof sketch. The counts and (i)–(iii) are one exhaustive evaluation over the 16 351 nonzero proper representatives of Proposition 4.2: for each, the reduced homology dimensions of all $2^6 = 64$ induced subcomplexes, and from them the profile and the Betti table, exactly as at $n = 5$. That evaluation runs over $\mathbb{F}_2$. Passing to an arbitrary field is Corollary 5.7 below: by Proposition 5.6 the only class whose graded Betti table depends on the characteristic is the class of the Stanley–Reisner ideal of the minimal six-vertex triangulation of $\mathbb{RP}^2$, and the exchange it causes changes neither count. Finally, the number of representatives, 16 353, and the number of nonzero proper ones are Proposition 4.2, so the left-hand end of the chain is not an artefact of the supplied list. □

The comparison with $n = 5$ is worth making explicit: the two losses are of the same order at $n = 6$, 8118 against 6629, whereas at $n = 5$ the summation dominated 55 to 19. The positional loss that [5, §10] names as the explanation of Betti table coincidences is thus a minority phenomenon at $n = 5$ and the larger of the two at $n = 6$.

Remark 4.4. The count 1604 was first obtained by a program written in C directly from the definitions, outside the formal development and independently of it, and that program agrees with Theorem 4.3 on every number stated there — including the fibre statistics — over $\mathbb{F}_2$ and over $\mathbb{F}_3$, and reports 1604 over $\mathbb{F}_{2147483647}$ as well. A content-hash comparison of its $\mathbb{F}_2$ and $\mathbb{F}_3$ runs finds exactly one relabelling class changing its table and its profile, the class of the $\mathbb{RP}^2$ ideal, whose canonical form is the bitmask 1708916285114496 and whose orbit has 12 members. The same program returns 16 351 labelled profiles for the 16 351 classes, the $n = 6$ instance of Proposition 3.3. None of this is part of the machine-checked development and none of it is used in any proof above; it is recorded as an independent cross-check.

5. Independence of the characteristic

Theorem 4.3 is stated over an arbitrary field, and at $n = 6$ that needs an argument, since six variables is exactly where characteristic dependence begins: not below six by [5, Thm. 1.1], and at six by Peeva's example [5, §7]. The tool is a rank certificate.

Definition 5.1. Let M be an $m \times n$ integer matrix. A *rank certificate of rank r* for M is a pair of integer matrices X $(r \times m)$ and V $(n \times r)$ with

$$X\,M\,V = I_r \qquad \text{and} \qquad (MV)(XM) = M.$$

Lemma 5.2. *Let R be a commutative ring, K a field and $f : R \to K$ a ring homomorphism. If $M \in R^{m \times n}$ admits X, V over R with $XMV = I_r$ and $(MV)(XM) = M$, then the matrix $f(M)$ has rank exactly r over K.*

Proof. Apply f entrywise. The first identity exhibits I_r as a product through $f(M)$, and rank does not grow along a product, so $r \le \operatorname{rank} f(M)$. The second writes $f(M)$ as a product of an $m \times r$ and an $r \times n$ matrix, so $\operatorname{rank} f(M) \le r$. □

Only the two integer identities are ever checked; how X and V were found never enters the argument, and a certificate exists exactly when the Smith normal form of M is $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$, that is when the cokernel is torsion free. We note that the weaker condition "some $r \times r$ minor is ± 1" is not equivalent: the matrix $(2\ \ 3)$ has rank 1 over every field and neither of its 1×1 minors is a unit.

The certificates are applied not to complexes but to *face sets*. For n, a cardinality c and a set Σ of c-subsets of $[n]$, let $\partial_c(\Sigma)$ be the integer matrix whose rows are indexed by Σ, whose columns are indexed by *all* $\binom{n}{c-1}$ subsets of cardinality $c-1$, and whose (F,G) entry is $(-1)^i$ when $G = F \setminus \{v\}$ with v the i-th element of F in increasing order, and 0 otherwise. Every one of the $2^{\binom{n}{c}}$ sets Σ is the cardinality-c face set of a complex on $[n]$ — of its own down-closure — so sweeping all of them, for every c, meets every complex on $[n]$ at once. At $n = 5$ this is $2^5 + 2^{10} + 2^{10} + 2^5 + 2 = 2114$ matrices instead of 7581 complexes; at $n = 6$ it is 2^{20} triangle sets and 65 666 face sets in the other five cardinalities, instead of 7 828 354 complexes.

Remark 5.3 (the zero-column step). Two matrices are in play, and identifying them is the one step of this paper that is argued rather than machine-checked. The sweep certifies $\partial_c(\Sigma)$, with all $\binom{n}{c-1}$ subsets as columns. The homology computation behind every graded Betti table and every homology profile uses instead the boundary matrix of the complex Δ itself, whose columns are only the $(c-1)$-*faces* of Δ. The two agree up to a permutation of columns and the adjunction of columns that are identically zero: if $G \subseteq F \in \Delta$ then $G \in \Delta$, so a column of $\partial_c(\Sigma)$ indexed by a non-face of Δ meets no row when Σ is the cardinality-c face set of Δ. Neither operation changes the rank over any field or the multiset of elementary divisors. Every step below that passes from a certified face set to a statement about complexes — Propositions 5.4, 5.5 and 5.6, and through them the "over every field" half of Theorem 4.3 and of Corollary 5.7 — uses this identification, and it is not part of the formal development.

Proposition 5.4. *All* 2114 *face sets on five vertices carry a verified rank certificate, as do all* 98 *on four and all* 18 *on three; and on every one of them the certified rank equals the value returned by each of the two independently written rank routines the present development uses — one bitset elimination over* $\mathbb{F}_2$*, one fraction-free integer elimination carrying the alternating signs.*

Granting Remark 5.3, Proposition 5.4 says that every simplicial complex on at most five vertices has torsion-free reduced integral homology and that every one of its boundary ranks is the same over every field. That is the fact [5, §10] states its Theorem 1.1 rests on, "that simplicial complexes on at most five vertices have torsion-free reduced integral homology"; here it is checked rather than cited. The rank agreement then makes the counts 8, 26, 134 of Section 6, the fibre data of Theorem 3.1, and the homology vectors that build $\mathcal{P}$ the values over an arbitrary field, and not merely over the two fields the development can compute in.

Proposition 5.5. *Of the* 2^{20} *sets of triangles on six vertices, exactly twelve carry no rank certificate, and those twelve are the* S_6*-orbit of the minimal six-vertex triangulation of* $\mathbb{R}\mathrm{P}^2$*. All* 65 666 *face sets of the other cardinalities on six vertices are certified. On the twelve exceptions the boundary matrix* ∂_3 *has rank* 10 *over* $\mathbb{Q}$ *and rank* 9 *over* $\mathbb{F}_2$*, and an explicit unimodular Smith normal form exhibits its elementary divisors as* $(1,1,1,1,1,1,1,1,1,2)$*; so, by Remark 5.3, a complex on* $[6]$ *whose faces of cardinality three are one of those twelve sets has torsion subgroup exactly* $\mathbb{Z}/2$ *in* $\widetilde{H}_1$*, and no torsion elsewhere.*

This is the homological content of Theorem 3.6 of Govc, Marzantowicz, Michalak and Pavešić [1], which [5, §7] quotes and which reads: "Let K be a connected simplicial complex on $n \leq 6$ vertices. Then K is either the unique minimal triangulation of $\mathbb{R}\mathrm{P}^2$ or is homotopy equivalent to a wedge of spheres." It is verified here by a different route, Smith normal forms of boundary matrices rather than fundamental groups. The sweep of 2^{20} triangle sets runs in sixteen blocks of 65 536.

Proposition 5.6. *Exactly* 12 *of the* 7 828 354 *squarefree monomial ideals in six variables — exactly one of the* 16 353 *relabelling classes, namely that of the Stanley–Reisner ideal of the minimal* $\mathbb{R}\mathrm{P}^2$ *triangulation — have a graded Betti table that depends on the characteristic of* k*. Every other one has the same graded Betti table over every field.*

Proof sketch. By Hochster's formula the graded Betti table of S/I depends on the characteristic of k exactly when some induced subcomplex of Δ_I has torsion in its reduced integral homology. For $|\sigma| \leq 5$ this is excluded by Proposition 5.4 with Remark 5.3, so only $\sigma = [6]$ can contribute, and by Proposition 5.5 the complexes on $[6]$ with torsion are exactly the twelve relabellings of the $\mathbb{R}\mathrm{P}^2$ triangulation. Each of the twelve exceptional triangle sets determines its complex: all fifteen edges are forced, and no 4-subset of $[6]$ has all four of its triples among the ten $\mathbb{R}\mathrm{P}^2$ triangles, so no tetrahedron can be added — both checked — giving the face vector $(1, 6, 15, 10)$ and hence exactly one ideal per triangle set. □

Corollary 5.7. *The set of graded Betti tables realized by the* 16 351 *nonzero proper classes in six variables is the same over every field, and has* 1604 *elements. The number of homology profiles is* 8233 *over every field, but the* set *of profiles is not the same: passing from characteristic* 2 *to any other characteristic removes one profile and adds one.*

Proof sketch. By Propositions 5.5 and 5.6, in characteristic other than 2 the data differs from the characteristic-2 data only at the $\mathbb{RP}^2$ class, whose table there is its table over $\mathbb{Q}$. One exhaustive pass over the 16 351 classes measures both sides of that exchange: 19 classes carry the characteristic-2 table of the $\mathbb{RP}^2$ class, so removing it leaves that table realized; and 83 classes already carry its $\mathbb{Q}$ table, so inserting it adds nothing. Hence the set of tables, and *a fortiori* the count 1604, is unchanged. The same pass shows the profile side behaves differently: exactly 1 class carries the characteristic-2 profile of the $\mathbb{RP}^2$ class, namely itself, and 0 classes carry its $\mathbb{Q}$ profile. So one profile leaves the set and one enters it, and the count 8233 survives by a cancellation rather than by an equality of sets. □

Corollary 5.7 makes no appeal to [5, Cor. 7.1], which states that for $S = k[x_1, \ldots, x_6]$ and a monomial ideal $M \subseteq S$ the multigraded, graded and total Betti numbers of S/M "are the same over all fields of characteristic different from 2". Thus, in their words, "any characteristic-dependence in six variables can only occur in characteristic 2". That corollary would give the passage between any two fields of characteristic other than 2, but not the comparison with characteristic 2 that the count needs, and it rests in turn on [1, Thm. 3.6], whose statement we quote above and whose homological consequence Proposition 5.5 establishes independently.

6. The published five-variable data, reproduced

All of the following are results of [5] or standard facts, reproduced here from the definitions before anything new was attempted; we list them because the counts of Sections 3 and 4 are computed by the same code path and would be worth nothing without them.

Proposition 6.1. *In three, four and five variables the numbers of squarefree monomial ideals, of relabelling classes, of nonzero proper classes and of distinct graded Betti tables are*

$$(20,\ 10,\ 8,\ 8), \qquad (168,\ 30,\ 28,\ 26), \qquad (7581,\ 210,\ 208,\ 134).$$

The five-variable line is [5, Prop. 8.1]. *Moreover the multiplicity table of the* 134 *types is that of* [5, §8]*:* 96 *tables are realized by one class,* 18 *by two,* 11 *by three,* 3 *by four,* 5 *by five and* 1 *by six, and the six classes sharing one table are exhibited explicitly. In three variables the graded Betti table is a complete invariant of the relabelling class; in four it is already not.*

Proposition 6.2. *For the squarefree Veronese ideal $I_{d,n} = (x_F : |F| = d)$ and its almost squarefree Veronese variant $I^-_{d,n}$ obtained by deleting one generator, the closed forms of* [5, §9]

$$\beta_{r-d+1,r}(S/I_{d,n}) = \binom{n}{r}\binom{r-1}{d-1}, \qquad \beta_{r-d+1,r}(S/I^-_{d,n}) = \binom{n}{r}\binom{r-1}{d-1} - \binom{n-d}{r-d}$$

hold for every $1 \le d \le n$ at $n = 5$, and the first also at $n = 4$. In particular [5, Ex. 9.1] *is reproduced: for $I_{3,5}$ the only nonzero graded Betti numbers are $\beta_{0,0} = 1$, $\beta_{1,3} = 10$, $\beta_{2,4} = 15$, $\beta_{3,5} = 6$. The almost Veronese formula also survives both degenerate ends $d = 1$ and $d = 5$, which* [5, §9] *does not remark on.*

Proposition 6.3. *Let $J \subseteq k[a, b, c, e, f, h]$ be Peeva's Example 12.4 as printed in* [5, §7]*,*

$$J = (abc,\ abf,\ ace,\ ahe,\ ahf,\ bch,\ bhe,\ bef,\ chf,\ cef).$$

Then J has the same canonical form under S_6 as the Stanley–Reisner ideal of the minimal six-vertex triangulation of $\mathbb{RP}^2$; that complex has face vector $(1, 6, 15, 10)$ and Euler characteristic 1*; and the nonzero graded Betti numbers of S/J are $\beta_{0,0} = 1$, $\beta_{1,3} = 10$, $\beta_{2,4} = 15$, $\beta_{3,5} = 6$ over $\mathbb{Q}$, and those together with $\beta_{3,6} = 1$ and $\beta_{4,6} = 1$ over $\mathbb{F}_2$ — exactly the two tables* [5, §7] *prints. The two tables differ* *only in the column $j = 6$. In five variables, by contrast, all 208 classes have the same graded Betti table over $\mathbb{F}_2$ as over $\mathbb{Q}$.*

Proposition 6.4. *For every n and every generating set, Δ_I is a simplicial complex, as is every induced subcomplex $\Delta_I|_\sigma$; the empty face lies in Δ_I if and only if I is proper, and hence for a proper ideal every induced subcomplex contains it. For every m and all functions f, r with $r(0) = r(m) = 0$,*

$$\sum_{k=0}^{m}(-1)^k\big(f(k) - r(k) - r(k+1)\big) = \sum_{k=0}^{m}(-1)^k f(k),$$

which read with f the face numbers and r the boundary ranks is the Euler–Poincaré identity for the computed homology dimensions; note that no hypothesis on what r is *is needed, only that it vanishes at both ends. Finally, on all 208 five-variable classes and all 6656 induced subcomplexes they contribute, the boundary maps compose to zero, the ranks at both ends vanish, no computed homology dimension is negative, row 0 of the graded Betti table is $\beta_{0,0} = 1$ and nothing else, and row 1 counts the minimal generators of I by degree.*

The last clause is the strongest of the consistency checks, because $\beta_{1,j}(S/I)$ is a fact about a minimal free resolution while the number of minimal generators of degree j is a fact about the antichain that mentions no homology; Hochster's formula is exactly what says they agree. The first three clauses and the Euler–Poincaré identity are proved in general, with no appeal to evaluation.

7. Controls

Each count above is accompanied by refutations of the neighbouring wrong answers and by checks that the hypotheses are doing work. We list them because two of them are traps a reader might otherwise fall into.

Proposition 7.1. *In five variables:*

(i) *dropping the degree -1 term from Hochster's formula returns 130 distinct graded Betti tables instead of 134. The whole discrepancy is one five-fold collapse: the five ideals annihilated are exactly the linear ideals $(x_1), (x_1, x_2), \dots, (x_1, \dots, x_5)$, the table of $S/(x_1, \dots, x_m)$ being the Koszul table whose only nonzero entries are $\beta_{j,j} = \binom{m}{j}$; every nonzero entry of such a table lies on the diagonal, every diagonal entry comes from a σ with $\Delta|_\sigma = \{\emptyset\}$, and so all five become the zero table at once. The multiplicity table records the arithmetic: 96 singletons become 91 and a new five-fold type appears, $134 - 5 + 1 = 130$.*

(ii) *Dropping the empty face from the complex outright — rather than dropping the degree -1 term — returns the published count 134, with all 208 tables wrong.*

(iii) *The restriction to nonzero proper ideals is not vacuous: keeping the zero and the unit ideal gives 136 tables, since the zero ideal contributes $\beta_{0,0} = 1$ alone and the unit ideal the identically zero table, and neither coincides with any proper ideal's table. Also $134 \notin \{130, 133, 135, 208\}$, and 208 is refuted concretely by the six-fold table of Proposition 6.1.*

Item (ii) is the reason the reproduction of Section 6 is stated as the multiplicity table and the row checks rather than as the headline number: matching 134 is not evidence that an implementation of Hochster's formula is correct.

Proposition 7.2. *The rank checker rejects a fabricated certificate for (2), a rank claimed one too small and a rank claimed one too large, and accepts a true certificate of rank 2 and one of rank 6 found on a genuine boundary matrix; the sweep that certifies all 2114 five-vertex face sets returns "not certified" on the six-vertex $\mathbb{R}\mathrm{P}^2$ complex, so five vertices is sharp by the same function that establishes it. The boundary maps compose to zero over $\mathbb{Z}$, not merely over $\mathbb{F}_2$, for every cardinality and every n from 2 to 6. The homology profile is a relabelling invariant, checked exhaustively at $n \leq 5$ — every class against every one of the $n!$ relabellings, 24 960 pairs and 798 720 induced-subcomplex homology computations at $n = 5$ — which is what makes an invariant computed at canonical representatives an invariant of the class. Summing a homology profile returns the graded Betti table on all 208 classes at $n = 5$ and on 4000* *representatives at $n = 6$, and the multiplicity table of* [5, §8] *comes back out of the profile computation unchanged. The whole homology vector of all* 6656 *five-variable induced subcomplexes agrees over* $\mathbb{F}_2$ *and over* $\mathbb{Q}$*, which is strictly stronger than the agreement of the summed tables, and the comparison is not one of two zero arrays. The packings used are lossless: no computed homology dimension is negative, the largest is* 6 *at* $n = 5$ *and* 10 *at* $n = 6$*, and the largest graded Betti number is* 20 *and* 45 *respectively. The counts* 189 *and* 1604 *are refuted at both neighbouring values, and the strict inequalities* $1604 < 8233 < 16\,351$ *and* $134 < 189 < 208$ *are checked rather than read off; the canonicality and orbit-size tests of Proposition 4.2 are non-vacuous; and the array sort introduced for* 16 353 *elements is checked to be the same function as the one it replaces.*

8. Verification

Every theorem, proposition, lemma and corollary of this paper is a theorem of a formal development written in Lean 4; *Mathlib* supplies the two parts that need a library — Lemma 5.2, which is stated in Mathlib's rank of a matrix, and the simplicial-complex model behind the general clauses of Proposition 6.4 — while the files carrying the enumerations import nothing at all. The development contains no `sorry` and no hand-declared axiom, and the repository reference is to be added at submission. Four things stated in this paper are not theorems of it, and each is flagged where it occurs. Hochster's formula is used as the definition of the graded Betti table and is not reproved (Section 2). Remark 5.3 identifies the boundary matrix the face-set sweep certifies with the one the homology computation solves; that identification is what turns the certificates of Propositions 5.4 and 5.5 into statements about complexes, and hence what carries the phrase "over every field" in Proposition 5.6, Theorem 4.3 and Corollary 5.7. The grouping displayed after Proposition 3.2 is read off the same evaluation as the block sizes the proposition states, but is not itself the content of a theorem. And Remark 4.4 reports a computation made outside the formal development, which no proof uses. Every count in this paper is machine-checked. The general statements — Lemma 5.2, the down-closure and empty-face statements and the Euler–Poincaré identity of Proposition 6.4 — are ordinary proofs and depend on no evaluation axiom. Everything else is finite and is carried by compiled evaluation through Lean's `native_decide`, which is what the exhaustive searches need: the enumeration of the 20, 168 and 7581 antichains and the 6656 induced subcomplexes at $n \leq 5$ (Theorem 3.1, Propositions 3.2, 3.3, 6.1, 6.2, 6.3, 6.4, 7.1); the 2114 and $65\,666 + 2^{20}$ face-set certificates (Propositions 5.4, 5.5, 5.6); the counting enumeration of $M(6) = 7\,828\,354$ antichains (Lemma 4.1); the $16\,353 \times 720$ canonicality and orbit computations (Proposition 4.2); and the pass over $16\,351 \times 64$ induced subcomplexes (Theorem 4.3, Corollary 5.7). Those statements depend on `propext`, `Quot.sound`, sometimes `Classical.choice`, and the per-declaration axiom that `native_decide` introduces; the projections off each bundled evaluation add none of their own, so for instance the whole of Theorem 3.1 rests on a single evaluation. Whole-development time is about eighty minutes on one loaded machine, dominated by the 2^{20} face-set sweep and by the six-variable pass; peak memory is about 1.6 gigabytes for the files that import no library at all and about 6.7 gigabytes for the two that import *Mathlib*. Three independent external checks validate the encoding rather than assume it: the enumerator reproduces the Dedekind numbers 20, 168, 7581, 7 828 354 [3, A000372] and the orbit counts 10, 30, 210, 16 353 [3, A003182]; the reduced homology is computed twice by independently written routines, one bitset elimination over $\mathbb{F}_2$ that sees no signs and one fraction-free integer elimination that carries the alternating signs of the simplicial boundary map, and they agree on every one of the 6656 five-variable induced subcomplexes; and the program of Remark 4.4, written from the definitions in a different language, agrees on every count at $n = 3, 4, 5, 6$.

9. Outlook

Two directions are open. The first is the second question of [5, §10], which structural families account for repeated Betti tables. Theorem 3.1 sharpens it by splitting the fibres in two: the 19 positional coincidences and the 55 coincidences of sums are different phenomena and deserve separate answers, and Proposition 3.2 gives the first witnesses of each. The second is $n = 7$, which is out of reach by the method used here: $M(7) = 2\,414\,682\,040\,998$, so the counting enumeration behind Lemma 4.1 alone would cost some $3 \cdot 10^5$ times what it costs at $n = 6$, and the torsion sweep of Section 5 would need

2^{35} face sets in each of two cardinalities. The enumerative question of [5, §10] needs a different idea past six variables. A cheaper intermediate is the profile chain for restricted classes at $n = 7$ — ideals generated in a single degree, say — where the enumeration is $2^{\binom{7}{d}}$ rather than a Dedekind number.

References

[1] D. Govc, W. Marzantowicz, Ł. P. Michalak and P. Pavešić, *Fundamental groups of small simplicial complexes*, arXiv:2511.02586 [math.AT] (4 November 2025). The numbering is that of v1, the only version; Theorem 3.6 is the statement quoted in Section 5, and its homological consequence is verified there independently of its proof.

[2] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics **227**, Springer, New York, 2005. Hochster's formula is Corollary 5.12 there, stated in reduced cohomology; [5, Thm. 4.4] quotes it in the homological form used in Section 2.

[3] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A000372 ("Dedekind numbers or Dedekind's problem: number of monotone Boolean functions of n variables, number of antichains of subsets of an n-set, …") and A003182 ("Dedekind numbers: inequivalent monotone Boolean functions of n or fewer variables, or antichains of subsets of an n-set"), the latter being the count of antichains up to permutation of the ground set — its own comment records that the labelled case is A000372. A003182 is the entry cited by [5] for its own orbit count 210 at $n = 5$.

[4] I. Peeva, *Graded syzygies*, Algebra and Applications **14**, Springer, London, 2011. Example 12.4 is quoted here only in the form printed in [5, §7], which is where we take both the generators and the two Betti tables of Proposition 6.3 from; we have not consulted the book itself.

[5] N. Ripke and P. Yoon, *Characteristic independence of Betti numbers of monomial ideals in five variables*, J. Pure Appl. Algebra **230** (2026), no. 10, Paper No. 108354; arXiv:2607.10639 [math.AC]. Every section, theorem, proposition, corollary and example number cited here is that of v2 (27 July 2026), whose comment records that the revision matches the published version; v1 (12 July 2026) and v2 agree verbatim on every statement quoted above.

[6] N. Ripke and P. Yoon, *Betti tables of squarefree monomial ideals in five variables*, the companion repository named in the data and code availability statement of [5]: `https://github.com/voltroom0606/fivevariablebettinumbers`. It holds the generating script, the Macaulay2 and Python computations and their output for $n = 5$; the 210 representatives and the 134 Betti table types are the only enumerative data in it.