A 2-ADIC OBSTRUCTION FOR BESSEL QUADRATURE FORMULAS WITH NODES IN $\mathbb{Q}(i) \cap S^1$

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Abstract. Let $w(z) = -e^{-2/z}/(4\pi i)$ be the weight on the unit circle for which the Bessel polynomials are orthogonal, and let $\mathcal{K} = \mathbb{Q}(i) \cap S^1$ be the set of rational points of the circle. Matsumura asks, in part (2) of his Problem 1.2, whether for a given r there is a quadrature formula of degree $r+1$ for w whose $r+1$ nodes all lie in $\mathcal{K}$. He answers *no* for $r = 1$ and for $r = 2$, the latter by exhibiting a genus-one curve of Mordell–Weil rank 0, and writes: "It seems hard to extend the proof of Theorem 1.4 to $r \geq 3$." We settle $r = 3$, in the negative, and the nodes are not assumed distinct. The proof is a count at the prime $1+i$ of $\mathbb{Z}[i]$. Writing each node as $\zeta/\bar{\zeta}$ with $\zeta = n + mi$ primitive and clearing denominators turns the degree-4 moment criterion into the single equation $F(\zeta) = 0$ in $\mathbb{Z}[i]$, where

$$F = 128 \prod n_j - 32 \prod m_j + 13 \prod \zeta_j + (8 - 10i) \prod (n_j - m_j) + (8 + 10i) \prod (n_j + m_j),$$

and the valuation of F at $1+i$ equals the number of j with n_j, m_j both odd, hence is at most 4; so $F \neq 0$. The same count re-proves the two published cases $r = 1$ and $r = 2$ with no elliptic curve and no computer algebra. We also record an explicit degree-4 formula with four distinct *rational* nodes, so that the obstruction is caused by the circle condition and not by the degree, and we measure, outside the formal development, exactly how far the obstruction reaches: it is confined to $r \leq 3$. Every theorem and proposition below has been formally verified in Lean 4 against *Mathlib*, with the four exceptions that the closing section lists; no statement rests on an exhaustive search.

1. Introduction

The Bessel polynomials (see [1])

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!} \left(\frac{x}{2}\right)^k$$

are orthogonal on the unit circle $S^1 \subset \mathbb{C}$ with respect to the weight

$$w(z) = -\frac{e^{-2/z}}{4\pi i}, \qquad \int_{S^1} w(z)\,dz = 1. \tag{1}$$

Following Matsumura [2], a *quadrature formula of degree n with m nodes* for w is a pair of tuples $x_1, \ldots, x_m \in \mathbb{C}$ (the weights) and $z_1, \ldots, z_m \in \mathbb{C}$ (the nodes) with

$$\sum_{k=1}^{m} x_k f(z_k) = \int_{S^1} f(z)\,w(z)\,dz \qquad \text{for every } f \in \mathbb{C}[z] \text{ with } \deg f \leq n. \tag{2}$$

Testing (2) on the monomials z^j, $j \leq n$, is equivalent to (2) itself, so a quadrature formula is exactly a solution of a finite moment system; and the moments are rational numbers, by the residue computation of [2, Remark 3.1]:

$$\mu_j := -\frac{1}{4\pi i} \int_{S^1} z^j e^{-2/z}\,dz = \frac{(-2)^j}{(j+1)!}, \qquad \mu = \Big(1,\, -1,\, \tfrac{2}{3},\, -\tfrac{1}{3},\, \tfrac{2}{15},\, -\tfrac{2}{45}, \ldots\Big). \tag{3}$$

Throughout this paper the closed form on the right of (3) is taken as the *definition* of μ_j, so that no integral occurs anywhere below and every argument is exact algebra over $\mathbb{Q}$, $\mathbb{Q}(i)$ and $\mathbb{Z}[i]$. The identity (3) itself is quoted from [2, Remark 3.1] and is not reproved here.

The question we take up concerns nodes with rational coordinates on the circle. Write

$$\mathcal{K} = \mathbb{Q}(i) \cap S^1 = \{\, a + bi : \ a, b \in \mathbb{Q},\ a^2 + b^2 = 1 \,\},$$

a dense subgroup of S^1. With m nodes a formula cannot have degree above $2m-1$ — the Stroud-type bound, stated in that form in the introduction of [2] and attributed there to [6, Proposition 4.3] and [7, Proposition 2.7] — and for *any* prescribed $r+1$ distinct nodes one can always achieve degree r by interpolation [2, Remark 2.3]; so with $r+1$ nodes the first genuinely constrained degree is $r+1$, and the extreme one is $2r+1$. Matsumura poses the following, which we quote verbatim from [2, Problem 1.2].

> (1) Does there exist a quadrature formula of degree $2r$ for Bessel polynomials with $r+1$ nodes on $\mathbb{Q}(\sqrt{-1}) \cap S^1$?
> (2) Does there exist a quadrature formula of degree $r+1$ for Bessel polynomials with $r+1$ nodes on $\mathbb{Q}(\sqrt{-1}) \cap S^1$?

Part (1) is not open: [2] answers it negatively for every $r \geq 1$ in its own Theorem 1.3, and for $r \geq 3$ it does so with the nodes allowed to lie anywhere in $\mathbb{Q}(i)$, not merely on the circle. We say this explicitly because the two parts are easy to conflate, and because at $r = 1$ they are literally the same question, $2r = r + 1$ there. Part (2) is the open one. Its state before the present paper is the following.

r	degree	nodes	answer	source
0	1	1	yes	not stated in [2]; immediate, see Prop. 6.1
1	2	2	no	[2, Theorem 1.3 (1)]: solve, and $\sqrt{11} \notin \mathbb{Q}$
2	3	3	no	[2, Theorem 1.4]: $\operatorname{rank} C(\mathbb{Q}) = 0$ in Magma
≥ 3	$r+1$	$r+1$	open	—

For $r = 2$ the proof in [2] reduces the moment system to the rational points of the genus-one curve $C : y^2 = -44x^4 - 84x^3 + x^2 + 84x + 43$ and computes that its Mordell–Weil rank is 0; the paper then closes the discussion with the sentence, again verbatim, "It seems hard to extend the proof of Theorem 1.4 to $r \geq 3$."

What is settled here. The first open case.

> *There is no quadrature formula of degree* 4 *for the Bessel weight* (1) *whose four nodes lie in* $\mathcal{K} = \mathbb{Q}(i) \cap S^1$. *The nodes are not assumed distinct.*

This is Theorem 5.1, the case $r = 3$ of [2, Problem 1.2 (2)]. The proof has three steps and no case analysis.

Step 1, the criterion. With $r+1$ nodes, exactness at degree $r+1$ is the single condition $\int_{S^1} \theta\, w = 0$ for $\theta(z) = \prod(z - z_k)$; this is the Riesz–Shohat criterion [2, Theorem 2.2], [5], specialised to $k = r+1$, and the direction we need is one line of linear algebra and needs no distinctness of the nodes. At $r = 3$ it reads

$$2 + 5e_1 + 10e_2 + 15e_3 + 15e_4 = 0 \tag{4}$$

in the elementary symmetric functions e_k of the four nodes (Theorem 2.2).

Step 2, the parameterisation. By Hilbert 90 for $\mathbb{Q}(i)/\mathbb{Q}$, a point $z \in \mathbb{Q}(i)$ with $z\bar{z} = 1$ satisfies $z\bar{\zeta} = \zeta$ for some primitive $\zeta = n + mi \in \mathbb{Z}[i]$ (Lemma 3.1); the witness is proportional to $1 + z$, and $\zeta = i$ at $z = -1$. Multiplying (4) by $\prod_j \bar{\zeta}_j$ clears it into a single equation $F(\zeta) = 0$ in $\mathbb{Z}[i]$.

Step 3, the descent. In closed form

$$F = 128 \prod_j n_j - 32 \prod_j m_j + 13 \prod_j \zeta_j + (8 - 10i) \prod_j (n_j - m_j) + (8 + 10i) \prod_j (n_j + m_j), \tag{5}$$

and, ζ_j being primitive, the valuation of F at the prime $1 + i$ equals $\#\{j : n_j, m_j \text{ both odd}\} \leq 4$, whereas each of the other four terms of (5) has strictly larger valuation. Hence $F \neq 0$ (Theorem 4.2).

The same three steps, run at $r = 1$ and $r = 2$, give the two published theorems again (Section 7): at $r = 1$ the criterion forces $\prod \zeta_j = 12 \prod n_j$, at $r = 2$ it forces 4 to divide both coordinates of $\prod \zeta_j$, and in each case a product of two, respectively three, primitive norms would have to be divisible by 16, which is impossible because a primitive sum of two squares has 2-adic valuation at most 1. So the whole of [2, Problem 1.2 (2)] below $r = 4$ is a single obstruction, and in particular the elliptic-curve computation of [2, Theorem 1.4] can be replaced by a two-line count. We record those two cases because a method that claims an open case ought first to reproduce the closed ones; they are the positive controls of this paper.

What the obstruction is not. It is not a statement about the degree. Over $\mathbb{Q}$ instead of $\mathcal{K}$ a degree-4 formula with four distinct rational nodes does exist — the nodes $0, 1, -1, -4/5$ with weights $-1/2$, $1/54$, $-5/6$, $125/54$, Proposition 6.1 — so it is the unit-circle condition that obstructs. That existence is an instance of [2, Theorem 1.5], which asserts a degree-$(r+1)$ formula with $r+1$ rational nodes for every $r \geq 0$ but prints one only at $r = 2$; the witness at $r = 3$ is ours. Nor is the criterion (4) empty on the circle: at the four points $1, -1, i, -i$ of $\mathcal{K}$ its left-hand side equals -13 (Proposition 6.2).

It also does not reach $r \geq 4$, and we say exactly where it stops. In the Cayley coordinate $z = (1+it)/(1-it)$ the criterion at $d = r+1$ nodes becomes two rational linear conditions on the elementary symmetric functions of $t_1, \ldots, t_d$, which pin the two coordinates of $\prod(1 + it_j)$; for $d \leq 4$ the pinned values carry a power of 2 large enough to beat the bound $v_{1+i}(\prod \zeta_j) \leq d$, and from $d = 5$ on they do not. This computation, and four exhaustive searches that returned nothing, are reported in Section 8; they lie outside the formal development and support no claim made here. The case $r = 4$ is open.

Provenance. The novelty claim attached to Theorems 4.2 and 5.1 is the outcome of a search, and is to be read as "not found", not as "new". What was searched, on 2026-08-30: the arXiv API reports v3 (2023-03-24) as the newest version of [2], and the abstract page lists no corrigendum; both citation graphs available to us — OpenAlex on the journal record and on the preprint record, and Semantic Scholar — return the same four citing works, of which two are unrelated papers in photometry and two are mathematical, namely [3] and [4]; both of the latter were fetched as e-prints and read in full, and neither addresses [2, Problem 1.2 (2)] at any $r \geq 3$. We note that [3], by the same author, contains a section on Bessel designs and closes with a *different* open question ("Does there exist an infinite family of designs of degree ≥ 3 for Bessel polynomials?"); that its author's own later paper poses a different Bessel question, and not this one, is positive evidence that he had not settled the one taken up here. A full-text search of a local arXiv mirror (367 shards, coverage through 2026-08-28) returned no occurrence of the identifier `2205.00464` or of the title of [2] in its mathematics shards — topical absence only, since that mirror drops bibliographies and is a weak citation index — and an arXiv metadata search for `abs:"Bessel polynomials" AND abs:"quadrature"` returns [2] and nothing else.

Two things we did not do. The version of record of [2] (Indagationes Mathematicae **34** (2023), no. 3, 622–636; the journal data are those of the publisher's Crossref record, not of the article) is behind a paywall and was not read. All citations to [2] in this paper are to the arXiv v3 e-print, and every problem, theorem, remark and example number we quote is the number carried by v3, checked against the e-print itself; whether the journal numbers them the same way we have not verified. The e-print postdates the journal's online publication date, so it is at least as new as the published text, and no corrigendum is listed. And the two papers of Sawa and Uchida [6, 7], cited in [2] as its `[SU19]` and `[SU20]` and described there as the Hermite, Legendre and Laguerre predecessors of its own results, were not read directly; every statement about them in this paper is quoted from the summary in [2] and is not load-bearing.

Plan. Section 2 fixes the moment system and proves the degree-4 criterion. Section 3 is the Gaussian parameterisation of $\mathcal{K}$. Section 4 contains the Diophantine core, Section 5 the main theorem and the degenerate cases, and Section 6 the controls. Section 7 re-proves the two published cases. Section 8 measures where the method stops, and Section 9 describes the formal verification.

2. The moment system and the degree-4 criterion

Definition 2.1. For $n, m \geq 0$, a pair of tuples $x, z \in \mathbb{C}^m$ is a *quadrature formula of degree n with m nodes* if

$$\sum_{k=1}^{m} x_k z_k^j = \mu_j \qquad \text{for all } 0 \leq j \leq n,$$

with $\mu_j = (-2)^j/(j+1)!$ as in (3). The nodes z_k are *not* required to be distinct, nor the weights x_k to be nonzero.

Dropping distinctness costs nothing in the positive direction — a formula with distinct nodes is a formula — and strengthens every non-existence statement, so we drop it throughout. The following is the direction of the Riesz–Shohat criterion that we use, specialised to four nodes and degree 4. In the notation below, $e_1, \dots, e_4$ are the elementary symmetric functions of z_1, z_2, z_3, z_4.

Theorem 2.2 (The degree-4 criterion)**.** *Let $x, z \in \mathbb{C}^4$ be a quadrature formula of degree 4 with four nodes. Then*

$$2 + 5e_1 + 10e_2 + 15e_3 + 15e_4 = 0.$$

Proof. Put $\theta(X) = \prod_{k=1}^{4}(X - z_k) = \sum_{j=0}^{4} c_j X^j$, so that $(c_0, c_1, c_2, c_3, c_4) = (e_4, -e_3, e_2, -e_1, 1)$. Since $\theta(z_k) = 0$ for every k,

$$0 = \sum_k x_k \theta(z_k) = \sum_{j=0}^{4} c_j \sum_k x_k z_k^j = \sum_{j=0}^{4} c_j \mu_j = e_4 + e_3 + \tfrac{2}{3}e_2 + \tfrac{1}{3}e_1 + \tfrac{2}{15},$$

where the third equality is Definition 2.1 at $j = 0, 1, 2, 3, 4$. Multiplying by 15 gives the statement. No hypothesis on the z_k beyond (2) is used; in particular they need not be distinct. □

Classically the criterion is an equivalence: for *distinct* nodes $z_1, \dots, z_{r+1}$ the weights are determined by interpolation and a degree-$(r+1)$ formula on those nodes exists if and only if $\int_{S^1} \prod(z - z_k)\, w = 0$. That is [2, Theorem 2.2], attributed there to Shohat [5]. Only the forward implication is needed here, and Theorem 2.2 is exactly that implication written out.

Remark 2.3. Before any use, the criterion was evaluated in exact rational arithmetic on each of the three formulas of degree $r+1$ with $r+1$ nodes that [2] prints, and returns 0 on each: the degree-3 formula with rational nodes $1/2$, $1/5$, $-27/44$ of [2, Example 5.1]; the degree-3 formula over $\mathbb{Q}(\sqrt{-3}) \cap S^1$ with nodes $-1, \omega, \bar{\omega}$ of [2, Remark 4.1 (2)]; and the unique degree-2 formula on S^1, with nodes $(-5 \pm \sqrt{-11})/6$, of [2, Remark 3.1]. For the first two, the full moment system was checked equation by equation for $j = 0, 1, 2, 3$, and it correctly fails at $j = 4$, as a degree-3 formula must. The one further formula printed in [2], that of its Example 3.2, has degree 4 with three nodes, that is degree $2r$ rather than $r+1$; it belongs to the case $k = 2$ of the Riesz–Shohat theorem and not to the case $k = r+1$ used here. These checks are reproductions of published examples, not claims of this paper.

3. Rational points of the circle

We write $\mathrm{N}(n + mi) = n^2 + m^2$ for the norm on $\mathbb{Z}[i]$, and call $\zeta = n + mi \in \mathbb{Z}[i]$ *primitive* if $\gcd(n, m) = 1$; a primitive ζ is in particular nonzero. Let $\lambda = 1 + i$, the prime of $\mathbb{Z}[i]$ above 2, and let $v = v_\lambda$ be the associated valuation, so $v(2) = 2$. We use repeatedly the elementary fact that

$$v(\zeta) = \begin{cases} 1, & n \text{ and } m \text{ both odd}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathrm{N}(\zeta) \equiv \begin{cases} 2 \pmod 4, & n \text{ and } m \text{ both odd}, \\ 1 \pmod 4, & \text{otherwise}, \end{cases} \tag{6}$$

for primitive $\zeta = n + mi$; so in either case the 2-adic valuation of the integer $\mathrm{N}(\zeta)$ equals $v(\zeta)$ and is at most 1. Indeed $\lambda \mid n + mi$ iff $n + m$ is even, and $\lambda^2 = 2i$ divides ζ only if $2 \mid n$ and $2 \mid m$, which primitivity forbids.

Lemma 3.1 (Hilbert 90 for $\mathbb{Q}(i)/\mathbb{Q}$)**.** *Let $z \in \mathcal{K}$, that is, $z = a + bi$ with $a, b \in \mathbb{Q}$ and $a^2 + b^2 = 1$. Then there is a primitive $\zeta = n + mi \in \mathbb{Z}[i]$ with*

$$z\,\bar{\zeta} = \zeta, \qquad \textit{that is,} \qquad z = \zeta/\bar{\zeta}.$$

Proof. If $1 + a = 0$ then $b = 0$, $z = -1$, and $\zeta = i$ works, since $-1 \cdot (-i) = i$. Otherwise put $t = b/(1 + a) \in \mathbb{Q}$ and let $(n, m) = (\operatorname{den} t, \operatorname{num} t)$ be the denominator and numerator of t in lowest terms; these are coprime by construction, so $\zeta = n + mi$ is primitive with no gcd division needed. From $m(1 + a) = bn$ and $a^2 + b^2 = 1$ one gets $an + bm = n$ and $bn - am = m$, and these two rational identities are the real and imaginary parts of $z\bar{\zeta} = \zeta$. □

The map is the classical rational parameterisation of the circle in disguise: in the Cayley coordinate $z = (1 + it)/(1 - it)$, the point z has parameter $t = b/(1 + a)$, and ζ is the primitive Gaussian integer $\operatorname{den}(t) + \operatorname{num}(t)\, i$. The value $t = \infty$, that is $z = -1$, is the case treated separately in the proof; keeping the multiplicative form $z\bar{\zeta} = \zeta$ rather than a fraction is what lets the node -1 be handled uniformly with the rest, and this matters, because with distinctness dropped several nodes may equal -1 at once.

4. The Diophantine core

Fix four primitive Gaussian integers $\zeta_j = n_j + m_j i$, $j = 1, 2, 3, 4$, and put

$$Q = \prod_j n_j, \quad P = \prod_j m_j, \quad R = \prod_j (n_j - m_j), \quad S = \prod_j (n_j + m_j), \quad U + Vi = \prod_j \zeta_j. \tag{7}$$

Let $F(\zeta) \in \mathbb{Z}[i]$ be the Gaussian integer obtained from the left-hand side of the criterion (4) at $z_j = \zeta_j/\bar{\zeta}_j$ by clearing denominators:

$$F(\zeta) = \sum_{T \subseteq \{1,2,3,4\}} c_{|T|} \prod_{j \in T} \zeta_j \prod_{j \notin T} \bar{\zeta}_j, \qquad (c_0, c_1, c_2, c_3, c_4) = (2, 5, 10, 15, 15). \tag{8}$$

Substituting $z_j = \zeta_j/\bar{\zeta}_j$ into $2 + 5e_1 + 10e_2 + 15e_3 + 15e_4$ and multiplying by $\prod_j \bar{\zeta}_j$ produces exactly (8), since e_k contributes the terms with $|T| = k$.

Lemma 4.1. *$F = 128Q - 32P + 13(U + Vi) + (8 - 10i)R + (8 + 10i)S$; that is, F has real part $128Q - 32P + 13U + 8R + 8S$ and imaginary part $13V - 10R + 10S$. Moreover $U^2 + V^2 = \prod_j \mathrm{N}(\zeta_j)$.*

Proof. Both are polynomial identities in the eight variables n_j, m_j and are proved by expansion. The first is obtained by writing F as a combination of $G(x) = \prod_j (\bar{\zeta}_j + x\zeta_j)$ at $x = \pm 1, \pm i$ together with one correction at $\prod \zeta_j$, using $\bar{\zeta} + i\zeta = (n - m)(1 + i)$ and $\bar{\zeta} - i\zeta = (n + m)(1 - i)$; the second is multiplicativity of the norm. □

Theorem 4.2 (The Diophantine core)**.** *Let $\zeta_j = n_j + m_j i$, $j = 1, 2, 3, 4$, be primitive Gaussian integers. Then the system*

$$128Q - 32P + 13U + 8R + 8S = 0, \qquad 13V - 10R + 10S = 0$$

in the notation (7) *has no solution. Equivalently, $F(\zeta) \neq 0$.*

Proof. We give the argument twice: first the valuation count, which is the reason, and then the elementary divisibility chain, which is what the formal development checks.

The count. Put $k = \#\{j : n_j, m_j \text{ both odd}\} \in \{0, 1, 2, 3, 4\}$, and read F in the closed form of Lemma 4.1. By (6), $v(\prod_j \zeta_j) = k$, so the term $13 \prod_j \zeta_j$ has valuation exactly k, since 13 is odd. For the others: $v(128Q) \geq v(128) = 14$ and $v(32P) \geq v(32) = 10$; and $n_j \mp m_j$ is even exactly for those j counted by k, while $v(8 \mp 10i) = v(2) + v(4 \mp 5i) = 2$, so the last two terms have valuation at least $2 + 2k$. Since $k \leq 4 < 10 < 14$ and $k < 2 + 2k$, the strict minimum of the five valuations is attained by the third term alone, whence $v(F) = k \leq 4$; in particular $F \neq 0$.

The chain. Suppose $F = 0$. The real equation reads $13U = 8\big(-16Q + 4P - R - S\big)$, and 13 is invertible modulo 8, so $8 \mid U$. The imaginary equation reads $13V = 10(R - S)$, so $2 \mid V$. Hence

$4 \mid U^2 + V^2 = \prod_j \mathrm{N}(\zeta_j)$ by Lemma 4.1. Each $\mathrm{N}(\zeta_j)$ is a primitive sum of two squares, hence 1 or 2 modulo 4 by (6), so its 2-adic valuation is at most 1; therefore at least *two* of the four norms are even. For those two indices j both $n_j - m_j$ and $n_j + m_j$ are even, so $4 \mid R$ and $4 \mid S$. Feeding this back into $13V = 10(R - S)$ gives $13V = 8 \cdot 5(R/4 - S/4)$ and hence $8 \mid V$. Now $U^2 + V^2$ is divisible by 64, so $\prod_j \mathrm{N}(\zeta_j)$ has 2-adic valuation at least 6; but it is a product of four positive integers each of valuation at most 1, so its valuation is at most 4. This contradiction proves the theorem. □

No search of any kind enters this proof. Two numerical checks were run beforehand, and are reported here only as a description of how the closed form was obtained and not as evidence for anything: the identity of Lemma 4.1 was verified on 30,000 random integer quadruples and on a box of 706,000 tuples before being proved, and the identity $v(F) = \#\{j : n_j, m_j \text{ both odd}\}$ on 200,000 random primitive quadruples, all observed pairs (k, v) lying on the diagonal from $(0, 0)$ to $(4, 4)$. An earlier and wrong closed form, obtained with $\bar{\zeta} + i\zeta$ misidentified, failed the first of these checks on 19,984 of 20,000 tuples.

5. Problem 1.2 (2) at $r = 3$

Theorem 5.1. *There is no quadrature formula of degree* 4 *for the Bessel weight with four nodes in* $\mathcal{K} = \mathbb{Q}(i) \cap S^1$. *That is, for every* $x \in \mathbb{C}^4$ *and every* $z_1, z_2, z_3, z_4 \in \mathcal{K}$, *not necessarily distinct, there is some* $j \le 4$ *with* $\sum_k x_k z_k^j \neq \mu_j$.

Proof. Suppose x, z were such a formula. By Theorem 2.2 the nodes satisfy $2+5e_1+10e_2+15e_3+15e_4 = 0$. By Lemma 3.1 choose primitive $\zeta_j = n_j + m_j i$ with $z_j\bar{\zeta}_j = \zeta_j$. Substituting $z_j\bar{\zeta}_j = \zeta_j$ into the definition (8) of F gives the polynomial identity

$$F(\zeta) = \big(2 + 5e_1 + 10e_2 + 15e_3 + 15e_4\big) \cdot \prod_j \bar{\zeta}_j = 0,$$

and by Lemma 4.1 the real and imaginary parts of $F(\zeta)$ are the two integer expressions of Theorem 4.2. That theorem forbids their simultaneous vanishing. □

Two features of the statement deserve to be isolated rather than left as asides, because they are what make it a complete answer to [2, Problem 1.2 (2)] at $r = 3$ and not merely to a generic case of it.

Proposition 5.2 (Degenerate node configurations)**.** *Theorem 5.1 assumes neither that the four nodes are distinct nor that the four weights are nonzero. Consequently there is no quadrature formula of degree* 4 *for the Bessel weight with* at most *four nodes in* $\mathcal{K}$*: a formula with* $m < 4$ *nodes in* $\mathcal{K}$ *becomes one with four nodes in* $\mathcal{K}$ *on adjoining* $4 - m$ *copies of the node* $1 \in \mathcal{K}$ *with weight* 0.

Proof. The first sentence is the hypothesis of Theorem 5.1 as stated, and the second is the padding just described, which changes no moment sum. □

Proposition 5.3 (The node -1)**.** *The point* -1 *lies in* $\mathcal{K}$ *and is covered by Theorem 5.1: it is the unique point of* $\mathcal{K}$ *whose Cayley parameter* $t = b/(1 + a)$ *is undefined, and Lemma 3.1 assigns to it the primitive Gaussian integer* $\zeta = i$. *In particular the theorem also excludes formulas in which one, two, three or all four nodes equal* -1.

Proof. Immediate from the first case of the proof of Lemma 3.1, together with Theorem 5.1, whose nodes are unconstrained points of $\mathcal{K}$. □

6. Controls

A non-existence theorem is worth only as much as the evidence that its hypotheses are not vacuous. We record four controls: two positive, showing that the objects exist once one of the two constraints is relaxed, and two negative, showing that neither the definition nor the criterion is satisfied by everything.

Proposition 6.1 (Positive controls). (1) *The nodes and weights*

$$(z_1, z_2, z_3, z_4) = \Big(0,\ 1,\ -1,\ -\tfrac{4}{5}\Big), \qquad (x_1, x_2, x_3, x_4) = \Big(-\tfrac{1}{2},\ \tfrac{1}{54},\ -\tfrac{5}{6},\ \tfrac{125}{54}\Big)$$

form a quadrature formula of degree 4 *for the Bessel weight, the four nodes being pairwise distinct and rational.*

(2) *The single node* -1 *with weight* 1 *is a quadrature formula of degree* 1*; that is,* [2, Problem 1.2 (2)] *is affirmative at* $r = 0$*, the node lying in* $\mathcal{K}$*.*

Proof. Both are evaluations of Definition 2.1: five rational identities $\sum_k x_k z_k^j = \mu_j$ for $j \le 4$ in the first case, two in the second. Distinctness in (i) is inspection. □

Part (i) is the case $r = 3$ of [2, Theorem 1.5], which asserts that a degree-$(r+1)$ formula with $r+1$ nodes in $\mathbb{Q}$ exists for every $r \ge 0$; that paper prints an explicit witness only at $r = 2$ (its Example 5.1), and the one above is ours, obtained by choosing three nodes and solving the criterion of Theorem 2.2 for the fourth, then solving the linear moment system for the weights. Together with Theorem 5.1 it locates the obstruction precisely: at degree 4 with four nodes, rationality of the nodes is compatible with existence and rationality *on the circle* is not. Part (ii) shows that [2, Problem 1.2 (2)] is not negative for trivial reasons; it is not stated in [2].

Proposition 6.2 (Negative controls). (1) *The formula of Proposition 6.1 (i) is* not *of degree* 5*: its moment identity fails at* $j = 5$*.*

(2) *The four points* 1, -1, i, $-i$ *lie in* $\mathcal{K}$*, and at them the left-hand side of the criterion* (4) *takes the value* -13*, not* 0*.*

Proof. (i) is one rational evaluation. In (ii), membership in $\mathcal{K}$ is witnessed by the pairs $(a, b) = (1, 0), (-1, 0), (0, 1), (0, -1)$, each with $a^2 + b^2 = 1$; and for these four nodes $e_1 = e_2 = e_3 = 0$ and $e_4 = -1$, so $2 + 5e_1 + 10e_2 + 15e_3 + 15e_4 = 2 - 15 = -13$. □

Control (i) says that Definition 2.1 is not vacuously satisfied at every degree, so that "degree 4" in Theorem 5.1 is a real constraint from above. Control (ii) says that the criterion (4) is a real constraint on quadruples drawn from $\mathcal{K}$, so that Theorem 4.2 is not proving something empty: there are quadruples of circle points at which the criterion visibly fails, and the content of Theorem 4.2 is that there are none at which it holds.

7. The published cases $r = 1$ and $r = 2$

The two cases of [2, Problem 1.2 (2)] already settled in [2] fall to the same three steps. We state them with their proofs because they are this method's controls against published theorems, and because the second replaces a Mordell–Weil rank computation by a divisibility count. In this section e_k denotes the elementary symmetric functions of the two, respectively three, nodes, and Q, P, R, S, U, V are as in (7) with the products taken over the corresponding index set.

Theorem 7.1 ($r = 1$; [2, Theorem 1.3 (1)]). *No quadrature formula of degree* 2 *for the Bessel weight has both of its two nodes in* $\mathcal{K}$*. More precisely: every quadrature formula of degree* 2 *with two nodes satisfies* $2 + 3e_1 + 3e_2 = 0$*; and for primitive* $\zeta_1, \zeta_2 \in \mathbb{Z}[i]$ *the system*

$$11n_1n_2 + m_1m_2 = 0, \qquad m_1n_2 + n_1m_2 = 0$$

has no solution.

Proof. The criterion is Theorem 2.2's argument at two nodes: with $\theta(X) = X^2 - e_1X + e_2$ one gets $e_2 + e_1 + \frac{2}{3} = 0$, and multiplying by 3 gives $2 + 3e_1 + 3e_2 = 0$. Clearing denominators by $z_j\bar{\zeta}_j = \zeta_j$ as in Section 4 produces $F_2 = 2\bar{\zeta}_1\bar{\zeta}_2 + 3(\zeta_1\bar{\zeta}_2 + \bar{\zeta}_1\zeta_2) + 3\zeta_1\zeta_2$, whose real and imaginary parts are the two displayed integer forms. Suppose both vanish. Then $\zeta_1\zeta_2 = (n_1n_2 - m_1m_2) + i(n_1m_2 + m_1n_2) = 12n_1n_2$, so

$$\mathrm{N}(\zeta_1)\mathrm{N}(\zeta_2) = \mathrm{N}(\zeta_1\zeta_2) = 144\,(n_1n_2)^2,$$

which is divisible by 16. The left-hand side is a positive integer, the ζ_j being nonzero; but by (6) each primitive norm has 2-adic valuation at most 1, so the product of two has valuation at most 2 — a contradiction. The first assertion follows by Lemma 3.1, exactly as in the proof of Theorem 5.1. □

Theorem 7.2 ($r = 2$; [2, Theorem 1.4])**.** *No quadrature formula of degree* 3 *for the Bessel weight has all three of its nodes in* $\mathcal{K}$*. More precisely: every quadrature formula of degree* 3 *with three nodes satisfies* $1 + 2e_1 + 3e_2 + 3e_3 = 0$*; and for primitive* $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{Z}[i]$ *the system*

$$36Q + R + S = 0, \qquad -4P - 3R + 3S = 0$$

has no solution.

Proof. With $\theta(X) = X^3 - e_1X^2 + e_2X - e_3$ one gets $-e_3 - e_2 - \frac{2}{3}e_1 - \frac{1}{3} = 0$, and multiplying by -3 gives $1 + 2e_1 + 3e_2 + 3e_3 = 0$. Clearing denominators produces $F_3 = \sum_{T\subseteq\{1,2,3\}} c'_{|T|} \prod_{j\in T} \zeta_j \prod_{j\notin T} \bar{\zeta}_j$ with $(c'_0, c'_1, c'_2, c'_3) = (1, 2, 3, 3)$, as in (8), and $2F_3$ has real part $36Q + R + S$ and imaginary part $-4P - 3R + 3S$. Now use the two expansion identities

$$2U = 4Q - R - S, \qquad 2V = -4P - R + S,$$

valid for any three Gaussian integers. If the system holds then $R + S = -36Q$, so $2U = 40Q$ and $U = 20Q$, hence $4 \mid U$; and $3(S - R) = 4P$, so $6V = 3(S - R) - 12P = -8P$, hence $3V = -4P$ and $4 \mid V$. Therefore 16 divides $U^2 + V^2 = \mathrm{N}(\zeta_1)\mathrm{N}(\zeta_2)\mathrm{N}(\zeta_3)$, a positive integer whose 2-adic valuation is at most 3 by (6) — a contradiction. Lemma 3.1 gives the first assertion as before. □

The two proofs and the proof of Theorem 4.2 are the same proof: the criterion forces $\prod_j \zeta_j$ to be divisible by a power of $1 + i$ that a product of primitive Gaussian integers cannot supply. It is worth naming what is thereby avoided. For $r = 2$, [2] reduces the moment system to the rational points of $C : y^2 = -44x^4 - 84x^3 + x^2 + 84x + 43$ and reports the computation $\operatorname{rank} C(\mathbb{Q}) = 0$ carried out in Magma; Theorem 7.2 replaces that by the two identities $2U = 4Q - R - S$, $2V = -4P - R + S$ and a count modulo powers of 2. For $r = 1$, [2] solves the 2×2 moment system explicitly and observes that the solution has $\operatorname{Im} z_1 = \pm\sqrt{11}/6 \notin \mathbb{Q}$; Theorem 7.1 never solves the system.

8. Where the obstruction stops

Everything in this section was computed outside the formal development, in exact rational arithmetic, and is reported as measurement. No statement elsewhere in this paper depends on it.

Remark 8.1 (The reach of the method). Parameterise $\mathcal{K}$ by the Cayley coordinate $z = (1+it)/(1-it)$, a bijection $\mathbb{Q}\cup\{\infty\} \to \mathcal{K}$ with $t = \infty \mapsto -1$, and let $s_1, \dots, s_d$ be the elementary symmetric functions of $t_1, \dots, t_d$. For $d = r + 1$ nodes the criterion becomes two rational linear conditions on the s_k, one on those of even index and one on those of odd index; since $\prod_j (1 + it_j) = \sum_k i^k s_k$, they pin the two coordinates of $\prod_j (1 + it_j)$. Computed exactly for $d = 2, \dots, 10$:

d	real coordinate	imaginary coordinate	bound $v(\prod \zeta_j) \le d$
2	12	0	2 beaten
3	20	$4 \cdot$ (one free parameter)	3 beaten
4	160/3	$(40/7)\, s_1$	4 beaten
5	$48 - (3/2)$(free)	$12(8s_1 + s_3)$	5 not beaten
6, 7, 8, 9, 10	2-content 4, 2, 4, 2, 4	2-content 4, 2, 4, 2, 4	not beaten

Writing $t_j = m_j/n_j$ in lowest terms, so that $1 + it_j = \zeta_j/n_j$, the $d = 4$ row reads $21 \prod_j \zeta_j = 40(28N + 3Ti)$ with $N = \prod n_j$ and $T = Ns_1 \in \mathbb{Z}$, and $v(40) = 6 > 4$; that is the descent of Section 4 in Cayley coordinates, and it is why the argument works at $d \le 4$. From $d = 5$ on the real coordinate acquires a half-integer coefficient on a free parameter and its valuation is unconstrained, and the largest 2-content available never again beats the bound d. The obstruction of this paper is therefore confined to $r \le 3$, and $r = 4$ needs a different idea. The $d = 2$ row reproduces the $\sqrt{11}$ of [2, Theorem 1.3 (1)] ($s_1 = 0$, $s_2 = -11$ force $t^2 = 11$), and the $d = 3$ row is the system that [2] pushes onto its genus-one curve.

Remark 8.2 (Searches that returned nothing). Four exhaustive searches were run, all in exact rational arithmetic and all empty. At $r = 3$: 9,727,204 distinct values of $p = 2\operatorname{Re} z$ in the conjugation-symmetric sub-family; a complete pair sweep over all 184 points of $\mathcal{K}$ of Cayley height at most 12, that is 16,836 pairs, after which the remaining node is confined to the intersection of a rational line with the circle; and 360,000 parameter pairs in the sub-family containing the node -1. At $r = 4$: a complete sweep over all triples of rational Cayley parameters of height at most 12 (183 values, $1.0 \cdot 10^6$ triples), fixing three parameters and solving the two conditions for the sum and product of the remaining two, whose discriminant must then be a rational square. The first of these searches was afterwards superseded by a proof: the conjugation-symmetric sub-family at $r = 3$ reduces to $(1 + t^2)(1 + w^2) = 160/3$, which has no rational solution because a primitive sum of two squares is never divisible by 3; and that sub-family is in any case contained in Theorem 4.2. The remaining three searches support nothing and are recorded only so that the $r = 4$ frontier is described honestly.

Remark 8.3 (The price of pushing $r = 4$ further). The $r = 4$ sweep is complete for any witness having three of its five Cayley parameters inside the box, and costs $O(H^6)$ triples at height H. At $H = 60$ (4,400 values, $1.4 \cdot 10^{10}$ triples) it would need a C implementation with 128-bit arithmetic, an estimated 3–6 CPU-hours. It would also not be decisive: the discriminant that must be a square is of size $\asymp H^{12}$, so it is a square with probability $\asymp H^{-6}$ against $\asymp H^6$ triples, and the expected number of hits per octave of H is $O(1)$. A bounded search is thus uninformative in either direction, which is why we stopped and looked for an argument instead.

9. Verification

Every theorem and proposition of Sections 2–7 has been formally verified in Lean 4 against *Mathlib* [8], with the exceptions listed at the end of this paragraph; the development contains no `sorry`, and every statement is checked by Lean's kernel and depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound` — in particular nothing here is delegated to compiled evaluation, and the tactic `native_decide` does not occur anywhere in the development. There is no exhaustive computation to delegate: the polynomial identities of Lemmas 4.1 and 3.1 and of Theorems 2.2, 7.1 and 7.2 are certified by ring normalisation, the divisibility steps by linear integer arithmetic, and the valuation bounds through Mathlib's `Nat.factorization`; the only case analysis in the descent is the choice of which two of the four norms are even in the proof of Theorem 4.2, six cases resolved by one lemma. Four points are not themselves machine-checked and are flagged here as such. The identification (3) of the moments with the contour integral is quoted from [2, Remark 3.1] and taken as the definition, so no integral is formalised. The first half of the proof of Theorem 4.2, the count in $v = v_{1+i}$, is done by hand; the formal development takes the equivalent elementary chain given as the second half, and it is that chain which is kernel-checked. Propositions 5.2 and 5.3 are restatements, and what they add to Theorem 5.1 and Lemma 3.1 — the padding by zero-weight nodes in the first, and the reading of the case distinction in the second — is done by hand. Finally, everything in Section 8 was computed outside the formal development, in C and in Python, and asserts nothing. Repository reference to be added at submission.

References

[1] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc. **65** (1949), 100–115. Not consulted directly: the normalisation of y_n and the orthogonality on S^1 recalled in Section 1 are quoted from [2], which cites pp. 101 and 104 of this paper for them; neither is used anywhere in the proofs.

[2] H. Matsumura, *Quadrature formulas for Bessel polynomials*, arXiv:2205.00464v3 (2023); Indag. Math. (N.S.) **34** (2023), no. 3, 622–636, doi:10.1016/j.indag.2023.01.001. All problem, theorem, remark and example numbers cited in this paper are those of the arXiv v3 e-print, which is the version we read; the journal data are from the publisher's Crossref record, the version of record itself being paywalled.

[3] H. Matsumura and M. Sawa, *Ellipsoidal designs and the Prouhet–Tarry–Escott problem*, arXiv:2502.17106v3; Ramanujan J. **68** (2025), no. 4, Paper No. 96.

[4] T. Mishima, X.-N. Lu, M. Sawa and Y. Uchida, *Geometric designs and Hilbert–Kamke equations of degree five for classical orthogonal polynomials*, arXiv:2503.21151v4; Res. Number Theory **12** (2026), no. 2, Paper No. 39.

[5] J. Shohat, *On mechanical quadratures, in particular, with positive coefficients*, Trans. Amer. Math. Soc. **42** (1937), no. 3, 461–496. Not consulted directly; it is the paper to which [2] attributes its Theorem 2.2, citing it there as `[Shohat, Theorem I]`.

[6] M. Sawa and Y. Uchida, *Discriminants of classical quasi-orthogonal polynomials with application to Diophantine equations*, J. Math. Soc. Japan **71** (2019), no. 3, 831–860.

[7] M. Sawa and Y. Uchida, *Algebro-geometric aspects of the Christoffel–Darboux kernels for classical orthogonal polynomials*, Trans. Amer. Math. Soc. **373** (2020), no. 2, 1243–1264. Neither this paper nor [6] was consulted directly; see the Provenance subsection of Section 1.

[8] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.