THE DIMENSION TEST FOR BALANCED HOLOMORPHIC VERTEX OPERATOR ALGEBRAS AT CENTRAL CHARGES 32 AND 40

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Abstract. Ampagouni, Mason and Mertens single out a class of holomorphic vertex operator algebras $V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus V_2 \oplus \cdots$ that they call *balanced*, for which the Virasoro vectors of V and of the subalgebra generated by V_1 coincide when the central charge is 32 or 40 and $V_1 \neq 0$, and they enumerate the 449 semisimple balanced root systems at $c = 32$ and the 1277 at $c = 40$. Three computational tests then try to rule each of them out. The cheapest is the *dimension test*: the integer $m = \dim V_2 - \dim L_{\widehat{\mathfrak{g}}}(k, 0)_2$ must be a non-negative integer combination of the top-level dimensions of the conformal-weight-two modules of $\widehat{\mathfrak{g}}$. At $c = 32$ exactly two root systems defeated even this test for want of resources, $A^3_{1,64}A_{4,160}$ and $A_{5,64}$; we decide both. The test *excludes* $A_{5,64}$, by a size obstruction ($m = 147519$ while every conformal-weight-two module has top-level dimension at least $3\,845\,961$, a minimum we determine exactly), and it does *not* exclude $A^3_{1,64}A_{4,160}$, where $m = 147094 = 6288 \cdot 23 + 10 \cdot 247$ is certified on the three $A_{1,64}$ factors alone. At $c = 40$ the source ran no test in its body, and its ancillary table records a result for 154 of the 1277 rows. We machine-check the census $\#\,\mathrm{BRS}(40, 0) = 1277$ (and find $\#\,\mathrm{BRS}(10, 0) = 22$ where the source prints 42), we show that the dimension test excludes 93 of the 1123 rows with no recorded result, and we show that one row the ancillary table marks as excluded by the dimension test, $A_{5,48}B_{2,24}$, is in fact not excluded by it. The $c = 40$ exclusions are stated about modules rather than about a computed list, through a completeness theorem for a pruned enumeration of the conformal-weight-two modules. A non-exclusion is not an existence statement, and an exclusion is an exclusion by this test only. All numbered statements are machine-checked in Lean 4; a displayed closed form for m in the source is also corrected.

1. Introduction

1.1. **Holomorphic vertex operator algebras and their weight-one Lie algebras.** A vertex operator algebra $V = \bigoplus_{n \geq 0} V_n$ of central charge c, with $V_0 = \mathbb{C}\mathbf{1}$, is *holomorphic* (or self-dual) if V is its own unique irreducible module. The weight-one space V_1 carries a Lie bracket $[a, b] = a_{(0)}b$, and when V_1 is reductive the Lie algebra structure together with the levels at which the affine algebras act is a strong invariant of V. For $c = 24$ the possible V_1 are Schellekens's celebrated list of 71 affine Kac–Moody structures [3], later given a uniform description by Höhn [4]; the classification programme that grew out of that list, and in particular the derivation of that list by van Ekeren, Möller and Scheithauer [5], is by now a large body of work.

Ampagouni, Mason and Mertens [1] carry the analogous programme to the next admissible central charges. They single out a class of holomorphic vertex operator algebras they call *balanced*, and show, in the words of their abstract, that "for a balanced, holomorphic VOA $V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus \cdots$ with $c = 32$ or 40 … the Virasoro vectors of V and the subVOA generated by V_1 coincide", which is what turns the classification into a finite root-system problem. Both notions are defined in the body of [1], and we use them exactly as defined there. For a root system $O^f\Phi_{1,k_1} \ldots \Phi_{n,k_n}$ with $c = f + \sum_i c_i$ and $d = f + \sum_i d_i$, [1, Definition 2] reads

> "We say that the root system $O^f\Phi_{1,k_1} \ldots \Phi_{n,k_n}$ is *balanced* if, for every index i, the following equality holds: $(c - f)\frac{h_i^\vee}{k_i} = d - c$."

and a vertex operator algebra is *balanced* when its own root system is ([1, Definition 10]: "We say that V is *balanced* if its associated root system lies in $B(c, f)$"), which for V with radical A they restate as $(c - \dim A)h_i^\vee/k_i = \dim V_1 - c$. The theorem quoted from the abstract is [1, Theorem 12], whose hypotheses are that V is balanced and holomorphic with $c = 32$ or 40 and $V_1 \neq 0$. Balancedness is a genuine hypothesis and not an automatic property at $c = 32$: [1] record that among the root systems of even unimodular lattices of rank 32, "exactly 13218 root systems occur and of these, just 82 are balanced".

In this paper V_1 is semisimple, so $f = \dim A = 0$, $d = d_1 := \dim V_1$, and c is 32 or 40. Writing

$$V_1 = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{r,k_r}$$

for the decomposition into simple summands with their levels, the displayed condition reads $c\,h_i^\vee = k_i(d_1 - c)$ for every i. In particular $h_i^\vee/k_i$ does not depend on i, and

$$k_i = \frac{c\,h_i^\vee}{d_1 - c}, \qquad \kappa_i := k_i + h_i^\vee = \frac{h_i^\vee\, d_1}{d_1 - c}, \tag{1}$$

so that a candidate is determined by the list of simple types alone; conversely, whenever every k_i in (1) is a positive integer one has $k_i/(k_i + h_i^\vee) = c/d_1$ and hence $\sum_i k_i \dim \mathfrak{g}_i/(k_i + h_i^\vee) = c$, so the list of simple types is a semisimple balanced root system of central charge c. Enumerating the possibilities gives, for $c = 32$, exactly 449 balanced root systems, the rows of [1, Appendix A], and for $c = 40$ exactly 1277, the rows of the ancillary table [2] (both counts are re-derived in §6). Every theorem below is a statement about *balanced* holomorphic vertex operator algebras only.

1.2. **The three tests, and where they stop.** Three computational tests are then applied to each of the 449 rows at $c = 32$: a *dimension test*, a *Jacobi forms test* and a *character test*. The source reports [1, §5.4] that

> "A total of 121 root systems were ruled out by the dimension test ..., 245 by the Jacobi forms test ..., and 25 by the character test .... The 19 root systems listed in Theorem 1 passed all three tests ... This leaves 38 root systems unaccounted for. For these, the computational resources the the [sic] authors' disposal were insufficient to finish the tests, either due to extremely long computation times (in some cases testing one root system was aborted after several months) and/or memory (exceeding 200 GB RAM)."

The 38 undecided rows are marked with a question mark in the appendix, and the appendix's legend says precisely how far the tests got on them:

> "A question mark ? next to a root system (38 cases) indicates that there was no conclusive result of the tests due to computational limitations: Except for the root systems $A_{1,64}^3 A_{4,160}$ and $A_{5,64}$, they all passed the dimension test, though, whereas for the two exceptions, even this relatively simple test could not be carried out using the computational ressources available to the authors."

So among the 449 rows there are exactly two on which the cheapest of the three tests has no recorded verdict. This paper supplies both verdicts.

At $c = 40$ the situation is different. The body of [1] runs no test there; its §5 says

> "The only thing that has prevented us extending this computation to the 1277 semisimple, balanced root systems with $c = 40$ is a dearth of computational resources."

The ancillary file [2], however, lists the 1277 semisimple balanced root systems of central charge 40 and says of them that "For the case $\dim A = 0$ we also record the available computational results on whether or not this root system can occur for a holomorphic VOA of central charge 40". Those recorded results are the label "X (d)" on 118 rows and the label "X (J)" on 36 rows, with no entry at all on the remaining 1123 rows. The ancillary file carries no legend of its own for its two labels; the legend of [1, Appendix A] explains the labels of the printed $c = 32$ table as

> "The letter X next to a root system means that the root system was ruled out. An additional note about the computational test that ruled out the root system – "dim" for the dimension test, "Jac" for the Jacobi forms test, and "char" for the character test (cf. Section 5) is also given."

and we read the ancillary's "X (d)" and "X (J)" as the abbreviations of that table's "X (dim)" and "X (Jac)". The three counts 118, 36, 1123 were taken from the text layer of the ancillary PDF, twice and independently; they are a reading of a document, not a machine-checked statement. Nothing below rests on them beyond this framing: every theorem of §§7–8 names its own rows, and the entry printed against each of those rows was read individually. Those 1123 blank rows are where this paper's $c = 40$ results live.

1.3. **Results at $c = 32$.** The dimension test asks whether an explicit integer m attached to a row is a non-negative integer combination of an explicit finite set D of module dimensions (§2). Our results are the following; $\omega_1, \ldots, \omega_n$ denote the fundamental weights and $\rho = \omega_1 + \cdots + \omega_n$.

- **$A_{5,64}$ is excluded** (Theorem 3.3). Here $\mathfrak{g} = \mathfrak{sl}_6$, $k = 64$, $m = 147519$, and every conformal-weight-two module has top-level dimension at least $3\,845\,961$. The obstruction is a pure size obstruction: the set D is not empty and no divisibility argument is used — every available part is simply larger than the target.
- **The minimum $3\,845\,961$ is exact** (Theorem 3.1), attained at $\lambda = \omega_1 + 4\omega_4 + 11\omega_5$, and every dominant weight of $\mathfrak{sl}_6$ with $\langle \lambda, \lambda + 2\rho \rangle = 280$ has level at most 19 — so the answer does not depend on the level being 64.
- **$A^3_{1,64}A_{4,160}$ is not excluded** (Theorem 4.1). Here $m = 147094 = 6288 \cdot 23 + 10 \cdot 247$, and the two parts come from conformal-weight-two modules supported on the three $A_{1,64}$ factors alone. The $A_{4,160}$ factor, whose $\binom{164}{4} = 29\,051\,001$ dominant weights of level at most 160 are what put the cell beyond reach, is never used. We stress that a non-exclusion is *not* an existence statement: the row remains open for the Jacobi forms and character tests, which the source could not run on it either.
- **A correction** (Proposition 2.3). The closed form for m displayed in [1, §5.1] has coefficient 1 where 247 is required; at the realised lattice theory $E^4_{8,1}$ the displayed expression is negative while the true value is 0. The definition of m and every conclusion drawn from it in [1] are unaffected.

Theorem 5.1 reproduces, through exactly the same machinery, a row the source itself decided by the dimension test, and Proposition 5.3 collects the negative controls. Remark 5.4 records an independent replay of the whole appendix, and is deliberately kept outside the formal development.

1.4. **Results at $c = 40$.**

- **The census** (Theorem 6.1). Every simple factor of a semisimple balanced root system of central charge c has $\dim \mathfrak{g} \le c(1+h^\vee)$, which makes the enumeration finite, and the counts $\#\,\mathrm{BRS}(c, 0)$ printed in [1, Table 4] are reproduced for $c = 1, \ldots, 9$ and $c = 16, 24, 32, 40, 48$ — in particular $\#\,\mathrm{BRS}(40, 0) = 1277$. The one exception is $c = 10$: the table prints 42, and the count is 22 (Proposition 6.2).
- **Three cells by hand** (Theorems 7.1, 7.2, 7.3). The dimension test excludes $A_{6,35}$ (a size obstruction, the minimum $194\,480$ determined exactly), $A^4_{3,1}G^{10}_{2,1}$ (a coin-problem obstruction that neither of the two criteria stated in [1, §5.1] decides) and $A_{40,1}$ (which has no conformal-weight-two module at all).
- **Ninety more exclusions** (Theorems 8.3–8.10). A generic engine — exact Lie data for the types A to G, a pruned enumeration of the conformal-weight-two modules with a machine-checked completeness theorem (Proposition 8.2), and the three obstruction routes — excludes 90 further rows of [2] with no recorded result, listed in Appendix A: 61 by the divisibility criterion of [1], 8 by a size obstruction, and 21 by a genuine coin-problem failure. Each exclusion is a statement about *modules*: no finite multiset of conformal-weight-two modules of the root system has total top-level dimension m.
- **A correction to the ancillary table** (Theorem 8.12). Row 93 of [2], $A_{5,48}B_{2,24}$, carries the entry "X (d)", but the dimension test does not exclude it: $m = 41\,860 = 88 \cdot 385 + 4 \cdot 1995$, and the two conformal-weight-two modules of dimensions 385 and 1995 are exhibited.

Throughout, "excluded" means excluded *by the dimension test*, as defined in §2 from the classical inputs listed there, and nothing more: no vertex operator algebra is constructed or destroyed beyond what the test says, and an excluded row is excluded exactly as the source's own "X (dim)" rows are.

1.5. **What makes the cells cheap.** The cost of the dimension test is the cost of computing D, and D is indexed by the dominant weights of conformal weight two. Enumerating dominant weights *by level* is what the source does, and at level 64 for A_5 there are $\binom{69}{5} = 11\,238\,513$ of them, at level 160 for A_4 there are $\binom{164}{4} = 29\,051\,001$, and for the four-factor row $A_{1,64}^3 A_{4,160}$ the product is about $8 \cdot 10^{12}$. The observation that removes this is that conformal weight two is a *norm* condition: it forces $\langle \lambda_i, \lambda_i + 2\rho_i \rangle \leq 4\kappa_i$ on every factor separately (Lemma 2.6), and in the normalisation $\langle \theta, \theta \rangle = 2$ the quadratic form $\langle \lambda, \lambda + 2\rho \rangle$ has non-negative coefficients in the Dynkin labels of λ and no constant term. Each label is therefore confined to an explicit finite range, and the enumeration becomes a box search: for A_5 at level 64, a box of 405 504 tuples with 80 solutions, all of level at most 19. For the four-factor row not even that is needed — a witness on three A_1 factors settles it. At $c = 40$ a second observation carries the remaining rows: since only parts of size at most m can enter a representation of m, and dimensions multiply across factors, a partial tensor product whose dimension already exceeds m can be discarded (§8).

2. The dimension test

Throughout, $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_i$ is a semisimple Lie algebra over $\mathbb{C}$ with each $\mathfrak{g}_i$ simple, $\widehat{\mathfrak{g}}_i$ is the untwisted affine algebra of $\mathfrak{g}_i$, $h_i^\vee$ is the dual Coxeter number, and θ_i the highest root. The invariant form is normalised by $\langle \theta_i, \theta_i \rangle = 2$ on each factor, so that $\langle \theta_i, \theta_i + 2\rho_i \rangle = 2h_i^\vee$. For a dominant integral weight λ of $\mathfrak{g}_i$ we write V_λ for the irreducible $\mathfrak{g}_i$-module of highest weight λ, $\operatorname{lev}(\lambda) = \langle \lambda, \theta_i^\vee \rangle$ for its level, and $L_{\widehat{\mathfrak{g}}_i}(k, \lambda)$ for the irreducible integrable highest-weight module of $\widehat{\mathfrak{g}}_i$ at level $k \geq \operatorname{lev}(\lambda)$. We use throughout the standard facts [6] that $L_{\widehat{\mathfrak{g}}_i}(k, \lambda)$ has conformal weight

$$h(\lambda) = \frac{\langle \lambda, \lambda + 2\rho_i \rangle}{2\kappa_i}, \qquad \kappa_i = k + h_i^\vee, \tag{2}$$

with top level V_λ, and that $\dim V_\lambda$ is given by the Weyl dimension formula.

2.1. **The integer m.** Let V be a balanced holomorphic vertex operator algebra of central charge $c \in \{32, 40\}$ with $V_1 = \mathfrak{g} = \bigoplus_i \mathfrak{g}_{i,k_i}$ semisimple, and let $U = \langle V_1 \rangle$, so that $U \cong \bigotimes_i L_{\widehat{\mathfrak{g}}_i}(k_i, 0)$, the conformal vectors of U and V agree, and V decomposes into finitely many irreducible U-modules with well-defined conformal weights. Since $V_0 = \mathbb{C}\mathbf{1}$ and $V_1 = \mathfrak{g}$ is the level-one part of the vacuum module already, no constituent other than U itself has conformal weight 0 or 1; hence

$$V_2 \;=\; L_{\widehat{\mathfrak{g}}}(k,0)_2 \;\oplus \bigoplus_{\operatorname{wt}(M)=2} m_M\, M_{\mathrm{top}}, \qquad m_M \in \mathbb{Z}_{\geq 0}, \tag{3}$$

where M runs over the irreducible U-modules of conformal weight 2. Consequently the integer

$$m := \dim V_2 - \dim L_{\widehat{\mathfrak{g}}}(k,0)_2$$

must be a non-negative integer combination of the finite set

$$D = \bigl\{ \dim M_{\mathrm{top}} : M \in \operatorname{Irr}(U),\ \operatorname{wt}(M) = 2 \bigr\}.$$

This is the *dimension test* of [1, §5.1], quoted there (for $c = 32$) as

> "Thus by Lemma 5 once more it must be possible to write the number
>
> $$m := \dim V_2 - \dim L_{\widehat{\mathfrak{g}}}(k,0)_2 = \dim V_1 + 139504 - \dim \operatorname{Sym}^2 \mathfrak{g}$$
>
> as a nonnegative integer linear combination of the dimensions of the irreducible modules of conformal weight 2 for the affine algebra. These dimensions can be computed based solely on the data associated with the root systems. Essentially the question becomes whether there is a partition of $\dim V_2$ whose parts are restricted to the (finite) set
>
> $$D = \{ \dim M : M \in \operatorname{Irr}(\widehat{V_1}),\ \operatorname{wt}(M) = 2 \}.$$
>
> Such a partition cannot possibly exist if $\gcd(D) \nmid m$ and it is guaranteed to exist if $m/\gcd(D) \in \mathbb{Z}$ exceeds the Frobenius number of $(1/\gcd(D)) \cdot D$ …"

The passage is quoted from v3, the current version, which carries no journal reference; the phrase "a partition of $\dim V_2$" is what v3 prints, and we read it as a partition of m, which is the number the preceding sentence asks to write as a combination and the number both criteria of the last sentence are about. If m is not such a combination, the balanced root system cannot occur, and the row is ruled out. We use the test at $c = 40$ in exactly the same form, with the graded dimension of V replaced by its $c = 40$ value below.

Definition 2.1. For $D \subseteq \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0}$ we say m is *D-representable* if there is a finite list $x_1, \ldots, x_s$ of elements of D (repetitions allowed, $s = 0$ allowed) with $x_1 + \cdots + x_s = m$.

2.2. **Computing m.** The two inputs are the graded dimension of V and the grade-two dimension of the vacuum module. The first is [1, Lemma 5]: for a holomorphic V of central charge 32, respectively 40,

$$Z_V = q^{-4/3}\big(1 + d_1 q + (248\, d_1 + 139504)q^2 + \cdots\big), \quad \text{so} \quad \dim V_2 = 248\, d_1 + 139504 \quad (c = 32), \tag{4}$$

$$Z_V = q^{-5/3}\big(1 + d_1 q + (496\, d_1 + 20620)q^2 + \cdots\big), \quad \text{so} \quad \dim V_2 = 496\, d_1 + 20620 \quad (c = 40). \tag{5}$$

The second is the standard Poincaré–Birkhoff–Witt count for the vacuum module: the grade-two subspace of the generalised Verma module of $\widehat{\mathfrak{g}}_i$ at level k_i is $\mathfrak{g}_i \oplus \operatorname{Sym}^2 \mathfrak{g}_i$, the singular vector $e_{\theta_i}(-1)^{k_i+1}\mathbf{1}$ that generates the maximal submodule sits in grade $k_i + 1$ ([6], the integrability relation at the affine node), and so for $k_i \geq 2$ the vacuum module has the same grade-two space, while for $k_i = 1$ one must remove the irreducible constituent $V_{2\theta_i}$ it generates. Adding the cross terms $\mathfrak{g}_i \otimes \mathfrak{g}_j$ of the tensor product gives

$$\dim L_{\widehat{\mathfrak{g}}}(k, 0)_2 = d_1 + \dim \operatorname{Sym}^2 \mathfrak{g} - \sum_{i:\, k_i = 1} \dim V_{2\theta_i}, \tag{6}$$

which is not stated in this form in [1]; the correction term is invisible in their §5.1, whose displayed formula presupposes every $k_i \geq 2$. Proposition 2.2 checks (6) in both regimes against the two $c = 32$ rows realised by lattice theories, where it can be read off a theta series instead, and Proposition 2.5 does the same at $c = 40$. We therefore have

$$m = 247\, d_1 + 139504 - \dim \operatorname{Sym}^2 \mathfrak{g} + \sum_{i:\, k_i = 1} \dim V_{2\theta_i} \qquad (c = 32), \tag{7}$$

$$m = 495\, d_1 + 20620 - \dim \operatorname{Sym}^2 \mathfrak{g} + \sum_{i:\, k_i = 1} \dim V_{2\theta_i} \qquad (c = 40), \tag{8}$$

with $\dim \operatorname{Sym}^2 \mathfrak{g} = \binom{d_1+1}{2}$. The three $c = 32$ rows treated in this paper have all levels at least 2, so for them the last sum is empty.

Proposition 2.2 (Values of m at $c = 32$)**.** *For the balanced root systems $A_{1,64}^3 A_{4,160}$, $A_{5,64}$ and $A_{6,14}$ of central charge* 32*, with $d_1 = 33, 35, 48$ respectively, formula* (7) *gives*

$$m\big(A_{1,64}^3 A_{4,160}\big) = 147094, \qquad m\big(A_{5,64}\big) = 147519, \qquad m\big(A_{6,14}\big) = 150184.$$

Moreover (7) *gives $m = 0$ at the two rows of* [1, Appendix A] *that are realised by lattice theories, namely $E_{8,1}^4$ with $d_1 = 992$ and $\dim V_{2\theta}(E_8) = 27000$, and $D_{32,1}$ with $d_1 = 2016$ and $\dim V_{2\theta}(D_{32}) = 1395680$.*

Proof. The levels come from (1): $32 \cdot 2/(33 - 32) = 64$ and $32 \cdot 5/(33 - 32) = 160$ for the first row, $32 \cdot 6/(35 - 32) = 64$ for the second, $32 \cdot 7/(48 - 32) = 14$ for the third, and $32 \cdot 30/(992 - 32) = 1$, $32 \cdot 62/(2016 - 32) = 1$ for the two lattice rows. Substituting d_1 and $\dim \operatorname{Sym}^2 \mathfrak{g} = \binom{d_1+1}{2}$ into (7) is then arithmetic; for instance $247 \cdot 35 + 139504 - 630 = 147519$. The two lattice rows are the independent check: V_2 of a holomorphic vertex operator algebra realised by a lattice theory whose V_2 is generated by V_1 must have $m = 0$, and indeed $992 + 492528 - 4 \cdot 27000 = 385520 = 248 \cdot 992 + 139504$ and $2016 + 2033136 - 1395680 = 639472 = 248 \cdot 2016 + 139504$. The value 385520 was also computed directly from the theta series of the lattice E_8^4, independently of (4): the grade-two dimension of a rank-32 lattice theory is $N_4 + 32N_2 + 560$ in terms of the numbers N_j of lattice vectors of norm j, and E_8^4 has $N_2 = 4 \cdot 240 = 960$ and $N_4 = 4 \cdot 2160 + \binom{4}{2} \cdot 240^2 = 354240$, giving $354240 + 30720 + 560 = 385520$. $\square$

Proposition 2.3 (A correction)**.** *The expression* $\dim V_1 + 139504 - \dim \mathrm{Sym}^2\,\mathfrak{g}$ *displayed in* [1, §5.1] *does not equal* $m = \dim V_2 - \dim L_{\widehat{\mathfrak{g}}}(k,0)_2$*; the coefficient of* $\dim V_1$ *should be* 247*, and at a row with a level-one factor the correction term of* (7) *is needed as well. At the row* $E_{8,1}^4$ *the displayed expression equals* $992 + 139504 - 492528 = -352032 < 0$*, whereas* $m = 0$ *there.*

Proof. Immediate from (4), (6) and Proposition 2.2: $\dim V_2 - \dim L_{\widehat{\mathfrak{g}}}(k,0)_2$ carries the coefficient $248 - 1 = 247$ on d_1. The inequality $992 + 139504 < \binom{993}{2} = 492528$ exhibits the failure numerically. □

Remark 2.4. Nothing in [1] depends on the displayed simplification. Their definition of m is the difference of dimensions, their tables are consistent with that definition (Remark 5.4), and no conclusion of theirs changes. We record the correction because the displayed expression is negative for every $d_1 \geq 529$ — one has $\binom{530}{2} = 140185 > 140033 = 529 + 139504$ — and so would be unusable as the target of a partition problem on the many rows of [1, Appendix A] with large d_1.

Proposition 2.5 (Values of m at $c = 40$)**.** *For the balanced root systems* $A_{6,35}$*,* $A_{3,1}^4 G_{2,1}^{10}$ *and* $A_{40,1}$ *of central charge* 40*, with* $d_1 = 48, 200, 1680$*, formula* (8) *gives*

$$m\big(A_{6,35}\big) = 43204, \qquad m\big(A_{3,1}^4 G_{2,1}^{10}\big) = 100626, \qquad m\big(A_{40,1}\big) = 179820,$$

using $\dim V_{2\theta}(\mathfrak{sl}_N) = N^2(N-1)(N+3)/4$*, which is* 84 *at* $N = 4$ *and* 739 640 *at* $N = 41$*, and* $\dim V_{2\theta}(G_2) = 77$*. Moreover* (8) *gives* $m = 0$ *at the two rows of* [2] *realised by lattice theories, row* 1274*,* $E_{8,1}^5$ *(*$d_1 = 1240$*,* $\dim V_{2\theta}(E_8) = 27000$*), and row* 1277*,* $D_{40,1}$ *(*$d_1 = 3160$*,* $\dim V_{2\theta}(D_{40}) = 3\,409\,560$*); and the level-one correction of* (6) *is not optional at* $c = 40$*: without it the row* $D_{40,1}$ *would receive* $m = 496 \cdot 3160 + 20620 - 3160 - \binom{3161}{2} = -3\,409\,560 < 0$*. Finally* $m = 0$ *is* D*-representable for every* D*, by the empty list, so the two lattice rows pass the test.*

Proof. The levels come from (1) with $c = 40$: $40 \cdot 7/8 = 35$; $40 \cdot 4/160 = 1$ for both A_3 and G_2; $40 \cdot 41/1640 = 1$; $40 \cdot 30/1200 = 1$; $40 \cdot 78/3120 = 1$. The rest is arithmetic in (8), for instance $495 \cdot 200 + 20620 - 20100 + (4 \cdot 84 + 10 \cdot 77) = 100626$ and $1240 + \binom{1241}{2} - 5 \cdot 27000 = 635660 = 496 \cdot 1240 + 20620$. The value $\dim V_2 = 496\,d_1 + 20620$ was re-derived independently of (5) from the theta series of E_8^5 and of the lattice D_{40}^+: the grade-two dimension of a rank-40 lattice theory is $N_4 + 40N_2 + 860$, giving $586800 + 48000 + 860 = 635660$ and $1462320 + 124800 + 860 = 1587980$ respectively, and two values determine the affine-linear relation between $\dim V_2$ and d_1 because the relevant space of modular forms is two-dimensional. The value $\dim V_{2\theta}(D_{40}) = 3\,409\,560$ was checked against $\mathrm{Sym}^2(\mathfrak{so}_{80}) = \Lambda^4\mathbb{C}^{80} \oplus V_{2\omega_2} \oplus \mathrm{Sym}_0^2\,\mathbb{C}^{80} \oplus \mathbb{C}$. □

2.3. **The norm bound.** The remaining ingredient is the description of D. An irreducible U-module is $M = \bigotimes_i L_{\widehat{\mathfrak{g}}_i}(k_i, \lambda_i)$ with each λ_i dominant integral of level at most k_i; by (2) its conformal weight is $\sum_i \langle \lambda_i, \lambda_i + 2\rho_i \rangle / (2\kappa_i)$ and its top level is $\bigotimes_i V_{\lambda_i}$.

Lemma 2.6. *If* $\mathrm{wt}(M) = 2$ *then* $\langle \lambda_i, \lambda_i + 2\rho_i \rangle \leq 4\kappa_i$ *for every* i*, with equality at* i *exactly when* $\lambda_j = 0$ *for every* $j \neq i$*; in particular with equality when* $r = 1$*. Moreover, for* $\mathfrak{g}_i$ *of type* A_n *and in the normalisation* $\langle \theta_i, \theta_i \rangle = 2$*, all the numbers* $\langle \omega_p, \omega_q \rangle$ *are positive, so* $\lambda = \sum_p a_p \omega_p \mapsto \langle \lambda, \lambda + 2\rho \rangle$ *is a quadratic form in* $a_1, \ldots, a_n$ *with positive coefficients and no constant term; the inequality therefore bounds each* a_p *individually.*

Proof. Each summand of $\sum_i \langle \lambda_i, \lambda_i + 2\rho_i \rangle / (2\kappa_i)$ is non-negative, and vanishes only for $\lambda_i = 0$, so each is at most the total 2 with equality only when the others vanish. For type A_n with $N = n + 1$ one has $\langle \omega_p, \omega_q \rangle = \min(p,q)\big(N - \max(p,q)\big)/N > 0$, so $N\langle \lambda, \lambda + 2\rho \rangle$ is an integer quadratic form in the a_p with strictly positive coefficients; discarding all terms but the ones involving a fixed a_p leaves $N\langle \omega_p, \omega_p \rangle a_p^2 + 2N\langle \omega_p, \rho \rangle a_p$, which already exceeds the target for a_p large. □

For $\mathfrak{g} = \mathfrak{sl}_6$ we write $c_5(\lambda) = 6\langle \lambda, \lambda + 2\rho \rangle$ explicitly in $\lambda = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4 + e\omega_5$:

$$\begin{aligned} c_5 &= (5a^2 + 30a) + (8b^2 + 48b) + (9c^2 + 54c) + (8d^2 + 48d) + (5e^2 + 30e) \\ &\quad + 2\big(4ab + 3ac + 2ad + ae + 6bc + 4bd + 2be + 6cd + 3ce + 4de\big), \end{aligned} \tag{9}$$

and for $\mathfrak{g} = \mathfrak{sl}_7$ we write $c_6(\lambda) = 7\langle\lambda, \lambda + 2\rho\rangle$ in $\lambda = a\omega_1 + \cdots + f\omega_6$ in the same way, its diagonal part being

$$(6a^2 + 42a) + (10b^2 + 70b) + (12c^2 + 84c) + (12d^2 + 84d) + (10e^2 + 70e) + (6f^2 + 42f). \tag{10}$$

The dimension of V_λ for $\mathfrak{sl}_{n+1}$ is the hook form of the Weyl formula,

$$\dim V_\lambda = \prod_{1\le p\le q\le n} \frac{a_p + a_{p+1} + \cdots + a_q + q - p + 1}{q - p + 1}, \tag{11}$$

whose denominator is the superfactorial $1!\,2!\cdots n!$, equal to 34560 for $n = 5$ and 24883200 for $n = 6$.

3. $A_{5,64}$: THE MINIMUM, AND THE EXCLUSION

The row $A_{5,64}$ has $\mathfrak{g} = \mathfrak{sl}_6$, $h^\vee = 6$, $k = 64$, $\kappa = 70$ and $d_1 = 35$, hence $m = 147519$ by Proposition 2.2. It is a single simple factor, so Lemma 2.6 gives conformal weight 2 exactly when $\langle\lambda, \lambda+2\rho\rangle = 4\kappa = 280$, that is $c_5(\lambda) = 1680$.

Theorem 3.1 (The minimal conformal-weight-two dimension)**.** *Let λ be a dominant integral weight of $\mathfrak{sl}_6$ with $\langle\lambda, \lambda + 2\rho\rangle = 280$. Then*

$$\dim V_\lambda \;\ge\; 3\,845\,961,$$

and the bound is attained, at $\lambda = \omega_1 + 4\omega_4 + 11\omega_5$. Moreover every such λ has level $\mathrm{lev}(\lambda) = a + b + c + d + e \le 19$. In particular integrability at level 64 imposes nothing beyond the Casimir equation: the set of conformal-weight-two highest weights of $A_5^{(1)}$ at level 64 is exactly the set of dominant λ with $c_5(\lambda) = 1680$.

Proof sketch. Every coefficient of (9) is a positive integer, so $c_5(\lambda) = 1680$ and $a \ge 16$ are incompatible: the two terms in a alone already give $5 \cdot 16^2 + 30 \cdot 16 = 1760 > 1680$. The same argument with $8b^2 + 48b$, $9c^2 + 54c$, $8d^2 + 48d$ and $5e^2 + 30e$ gives $b \le 11$, $c \le 10$ (here $9 \cdot 11^2 + 54 \cdot 11 = 1683 > 1680$, so the bound is tight by three), $d \le 11$, $e \le 15$. The solution set therefore lies in the box $[0, 15]\times[0, 11]\times[0, 10]\times[0, 11]\times[0, 15]$ of 405 504 tuples, and the remainder of the proof is the exhaustive evaluation of (9) and (11) at every one of those 405 504 tuples. The search certifies three things simultaneously for each tuple satisfying $c_5 = 1680$: that the product in (11) is divisible by 34560, so that no divisibility is assumed anywhere; that the quotient is at least 3 845 961; and that $a+b+c+d+e \le 19$. Attainment is a single evaluation, $\dim V_{\omega_1+4\omega_4+11\omega_5} = 3\,845\,961$; sharpness follows since the same witness refutes the bound 3 845 962. The statement about the level is then immediate, because $19 \le 64$ makes integrability automatic. □

Remark 3.2. The following further facts are read off the same exhaustive search; they are recorded for orientation and are *not* part of the machine-checked statement of Theorem 3.1, which certifies the lower bound, the attainment at $\omega_1 + 4\omega_4 + 11\omega_5$ and the level bound only. The search finds exactly 80 dominant weights with $c_5(\lambda) = 1680$, of which 2 are fixed by the diagram automorphism ($2\omega_2+8\omega_3+2\omega_4$ and $\omega_1 + 5\omega_2 + 2\omega_3 + 5\omega_4 + \omega_5$) and the remaining 78 fall into 39 interchanged pairs; the minimum is attained on one such pair, at $\omega_1 + 4\omega_4 + 11\omega_5$ and at $11\omega_1 + 4\omega_2 + \omega_5$. The six smallest distinct values of $\dim V_\lambda$ over the 80 are 3 845 961, 11 223 927, 12 899 250, 13 250 952, 14 965 236 and 19 045 125, the second occurring at $\lambda = \omega_3 + 8\omega_4 + 5\omega_5$.

Theorem 3.3 ($A_{5,64}$ is excluded)**.** *Let D be the set of top-level dimensions of the conformal-weight-two irreducible modules of $A_5^{(1)}$ at level 64. Then $m = 147519$ is not D-representable. Consequently the dimension test of* [1, §5.1] *rules out the balanced root system $A_{5,64}$ at central charge 32: no balanced holomorphic vertex operator algebra of central charge 32 has weight-one Lie algebra $\mathfrak{sl}_6$ at level 64.*

Proof. By Theorem 3.1 every element of D is at least 3 845 961. A D-representation of m is a finite list of elements of D summing to m. The empty list sums to $0 \ne 147519$, and a nonempty list has a first entry $x \in D$ with $x \le \sum = 147519 < 3\,845\,961 \le x$, a contradiction. □

Remark 3.4. The shape of this obstruction is worth isolating. The source states two criteria for the partition problem: a representation cannot exist if $\gcd(D) \nmid m$, and is guaranteed to exist once $m/\gcd(D)$ exceeds the Frobenius number of $\gcd(D)^{-1}D$. The proof above uses neither. The set D is not empty — Theorem 3.1 exhibits an element — and no divisibility is invoked; the exclusion is purely a size obstruction, $\min D > m > 0$. What matters practically is that both of the source's criteria presuppose that D has already been computed, and computing D is exactly the step that was out of reach; for an enumeration organised by level, a size obstruction is invisible until the enumeration finishes. The obstruction is also robust in the sense of Theorem 3.1: what cuts D out is the Casimir equation $\langle\lambda, \lambda + 2\rho\rangle = 280$, and the integrability bound at level 64 is never active, every solution having level at most 19.

This is not a claim that the source's divisibility criterion fails at this row. Once D has been computed one finds every one of its elements divisible by 81, so $\gcd(D) = 81$, and $147519 = 81 \cdot 1821 + 18$; the criterion $\gcd(D) \nmid m$ therefore excludes $A_{5,64}$ as well. What the size obstruction buys is not a stronger conclusion but a cheaper one: it needs only a lower bound for the elements of D, whereas $\gcd(D)$ needs D itself, and D was the object out of reach. The value $\gcd(D) = 81$ is a by-product of the same unformalised search as Remark 3.2; it is not asserted by Theorem 3.3, whose proof is the size comparison alone.

4. $A_{1,64}^3 A_{4,160}$: NOT EXCLUDED

The row $A_{1,64}^3 A_{4,160}$ has $d_1 = 3 \cdot 3 + 24 = 33$, levels 64 on each A_1 factor and 160 on the A_4 factor by (1), and $m = 147094$ by Proposition 2.2. All four levels exceed 1, so (7) has no correction term.

On a single A_1 factor at level 64 we have $\kappa = 66$, the dominant weights are $\lambda = n\omega$ with $0 \le n \le 64$, and (2) reads $h(n\omega) = n(n+2)/264$ with $\dim V_{n\omega} = n+1$. A module supported on the three A_1 factors, trivial on the A_4 factor, therefore has conformal weight 2 exactly when

$$n_1(n_1+2) + n_2(n_2+2) + n_3(n_3+2) = 528, \qquad 0 \le n_i \le 64, \tag{12}$$

and then top-level dimension $(n_1+1)(n_2+1)(n_3+1)$.

Theorem 4.1 ($A_{1,64}^3 A_{4,160}$ is not excluded)**.** *Let D be the set of top-level dimensions of the conformal-weight-two irreducible modules of the affine algebra attached to $A_{1,64}^3 A_{4,160}$, that is of $\mathfrak{sl}_2$ at level 64 on three factors and $\mathfrak{sl}_5$ at level 160 on the fourth. Then $23 \in D$ and $247 \in D$, and*

$$m = 147094 = 6288 \cdot 23 + 10 \cdot 247,$$

so m is D-representable. The dimension test of [1, §5.1] *does not rule out the balanced root system $A_{1,64}^3 A_{4,160}$ at central charge* 32.

Proof. The triple $(n_1, n_2, n_3) = (22, 0, 0)$ satisfies (12), since $22 \cdot 24 = 528$, and gives top-level dimension $23 \cdot 1 \cdot 1 = 23$; the triple $(0, 12, 18)$ satisfies it as well, since $12 \cdot 14 + 18 \cdot 20 = 168 + 360 = 528$, and gives $1 \cdot 13 \cdot 19 = 247$. Both have all $n_i \le 64$, so both are integrable at level 64. The list consisting of 6288 copies of 23 and 10 copies of 247 has sum $144624 + 2470 = 147094 = m$. □

Remark 4.2. Three points about Theorem 4.1. First, *a non-exclusion is not an existence statement.* The theorem says only that this particular test does not rule the row out; no vertex operator algebra is constructed, and the row remains among the open ones, now for the Jacobi forms and character tests — which the source could not run on it either. Second, the certificate never touches the $A_{4,160}$ factor: its $\binom{164}{4} = 29\,051\,001$ dominant weights of level at most 160, and the $65^3 \cdot 29\,051\,001 \approx 8 \cdot 10^{12}$ four-factor combinations they generate, are precisely what put the row out of reach, and none of it is needed to answer the question the test asks. Third, the certificate is a *witness*, so it is insensitive to any incompleteness in the computation of D: enlarging D can only preserve representability. This is the opposite situation to Theorem 3.3, where the conclusion does depend on D having been computed in full, which is why the enumeration there is exhaustive over a box that Lemma 2.6 proves to contain every solution.

Remark 4.3. For orientation, and again outside the formal development — Theorem 4.1 uses only the two parts 23 and 247 — an enumeration of the whole of D for this row returns 1818 distinct dimensions, of which 13 are at most m: 23, 247, 567, 1463, 2025, 2057, 4536, 18360, 32130, 36936, 55566, 92610, 113274. Several representations of m exist, but none uses a single repeated part: $147094 = 2 \cdot 73547$ with 73547 prime, so its divisors are 1, 2, 73547 and 147094, and none of them lies in D. The certificate above is therefore minimal in the number of distinct parts, and both of its parts are needed, since $23 \nmid 147094$.

5. A decided row at $c = 32$, and controls

Theorem 5.1 (Control: $A_{6,14}$)**.** *Let D be the set of top-level dimensions of the conformal-weight-two irreducible modules of $A_6^{(1)}$ at level* 14*. Every element of D that is at most* 150184 *is divisible by* 1716*, and* $1716 \nmid 150184$*; hence $m = 150184$ is not D-representable and the dimension test rules out the balanced root system $A_{6,14}$ at central charge* 32*.*

Proof sketch. Here $\mathfrak{g} = \mathfrak{sl}_7$, $h^\vee = 7$, $k = 14$, $\kappa = 21$, so conformal weight 2 means $\langle \lambda, \lambda + 2\rho \rangle = 84$, that is $c_6(\lambda) = 588$. The coefficientwise argument of Lemma 2.6 applied to (10) gives $a \leq 7$, $b, c, d, e \leq 4$, $f \leq 7$, a box of 40 000 tuples, over which the search is exhaustive; it certifies for every solution that the Weyl quotient (11) is exact, that $\dim V_\lambda > 150184$ or $1716 \mid \dim V_\lambda$, and that $\mathrm{lev}(\lambda) \leq 7 \leq 14$. One solution is $\lambda = 7\omega_6$, with $\dim V_\lambda = 1716$, so D is not empty. Every part usable in a representation of m is at most m, hence divisible by 1716, hence so is any sum of parts; but $150184 = 1716 \cdot 87 + 892$. ☐

Remark 5.2. Theorem 5.1 decides nothing new: the row $A_{6,14}$ is one of those [1, Appendix A] already rules out by the dimension test. It is included because it is a check on the machinery of §2 against a published verdict, run through the same box search and the same representability definition as Theorem 3.3, and because it exercises the other obstruction — a divisibility obstruction of the kind the source names, rather than the size obstruction of Remark 3.4.

Proposition 5.3 (Controls at $c = 32$)**.** *The following hold.*

(1) *The bound of Theorem 3.1 is sharp:* 3 845 962 *is not a lower bound for* $\dim V_\lambda$ *over the dominant λ of $\mathfrak{sl}_6$ with $\langle \lambda, \lambda + 2\rho \rangle = 280$.*
(2) *The failure in Theorem 3.3 is a property of the target and not of D: the number* 3 845 961 is *D-representable for the same D.*
(3) *Both parts of the certificate of Theorem 4.1 are needed: $23 \nmid 147094$, and $23p + 247q = 100$ has no solution in non-negative integers.*
(4) *The conformal-weight condition* (12) *is selective: the neighbouring A_1 weights $n = 21$ and $n = 23$ do not satisfy it.*
(5) *The two formulas are calibrated: for $\mathfrak{sl}_6$, $c_5(\omega_1 + \omega_5) = 72 = 6 \cdot 2h^\vee$ and* (11) *returns* 35*,* 6*,* 15*,* 20 *at $\omega_1 + \omega_5$, ω_1, ω_2, ω_3; for $\mathfrak{sl}_7$, $c_6(\omega_1 + \omega_6) = 98 = 7 \cdot 2h^\vee$ and* (11) *returns* 48*.*
(6) *Conformal-weight-two modules exist in all three rows: D contains* 3 845 961 *for $A_{5,64}$,* 23 *and* 247 *for $A_{1,64}^3 A_{4,160}$, and* 1716 *for $A_{6,14}$, so none of the statements above is vacuous.*

Proof. Each item is a finite evaluation or a two-variable linear Diophantine check; (1) and (2) use the witness $\omega_1 + 4\omega_4 + 11\omega_5$ of Theorem 3.1, and (6) uses the witnesses of Theorems 3.1, 4.1 and 5.1. ☐

Remark 5.4 (An independent replay of the appendix; outside the formal development)*.* To calibrate the reading of [1, §5.1] adopted in §2 we re-implemented the dimension test from scratch, in exact rational arithmetic — Cartan matrices for types A through G with the correct short-root scalars, positive roots by reflection closure, $\langle \omega_p, \omega_q \rangle$ from the inverse Cartan matrix, and the Weyl dimension formula, validated by $\dim V_\theta = \dim \mathfrak{g}$, $\langle \theta, \theta + 2\rho \rangle = 2h^\vee$ and $\mathrm{lev}(\theta) = 2$ on $A_1, A_5, B_2, B_4, C_3, C_4, D_4, D_{32}, G_2, F_4, E_6, E_7, E_8$ — and ran it on all 449 rows of [1, Appendix A], whose levels all satisfy (1). Excluding the two rows treated above, 443 of the remaining 447 verdicts agree with the printed labels. All 36 other undecided rows pass the dimension test, as the source states. The four disagreements are, by the row numbers of the printed table: rows 134 ($B_{4,7} G_{2,4}^2$) and 403 ($A_{2,1}^{10} B_{2,1}^2 D_{4,2}$), labelled as ruled out by the dimension test, which our replay finds representable (for the second, the usable parts are $\{675, 729, 875, 945\}$ and

$163204 = 675 \cdot 207 + 729 + 875 \cdot 26$); and rows 291 ($A_{5,4}^2 B_{2,2}$) and 303 ($A_{1,1}^6 E_{6,6}$), labelled as ruled out by the Jacobi forms test, which our replay already rules out by the dimension test — the same verdict under a different attribution. We also count 122 rows carrying the dimension-test label in the printed table, against the 121 of the prose quoted in §1; the totals $122 + 245 + 25 = 392$ and $122 + 245 + 25 + 38 + 19 = 449$ are consistent with the rest of the source's tally. This replay is ordinary floating-point-free computer algebra and is *not* part of the formal development of §9; we record it as a measurement, and we do not assert which side of the four disagreements is correct. Deciding that needs the authors' own code, which is not published. Neither of the two rows decided in this paper is exposed to the question: $A_{5,64}$ has a single simple factor, so no tensor combinations arise at all, and the certificate of Theorem 4.1 is a witness, which an incomplete D can fail to find but can never falsely produce. Row numbers here and throughout are those printed in the first column of the appendix table of v3: the two cells decided here are rows 2 ($A_{1,64}^3 A_{4,160}$) and 26 ($A_{5,64}$), the control of Theorem 5.1 is row 72, and the two lattice rows of Proposition 2.2 are rows 447 ($E_{8,1}^4$) and 449 ($D_{32,1}$).

6. The census of balanced root systems

By (1) and the sentence following it, a semisimple balanced root system of central charge c is the same thing as a finite multiset of simple types $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ with $d_1 = \sum_i \dim \mathfrak{g}_i > c$ such that every $k_i = c\, h_i^\vee/(d_1 - c)$ is a positive integer. [1, Table 4] prints the number $\#\,\mathrm{BRS}(c, 0)$ of these for $c = 1, \ldots, 10$ and $c = 16, 24, 32, 40, 48$; the entry 1277 at $c = 40$ is also the number of rows of [2].

Theorem 6.1 (The census)**.** (1) *Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ be a semisimple balanced root system of central charge c. Then every factor satisfies $\dim \mathfrak{g}_i \le c\,(1 + h_i^\vee)$.*

(2) *The number of semisimple balanced root systems of central charge c is*

$$\#\,\mathrm{BRS}(c, 0) = 1,\ 3,\ 3,\ 7,\ 8,\ 13,\ 15,\ 16,\ 20 \quad \textit{for } c = 1, \ldots, 9,$$
$$\#\,\mathrm{BRS}(c, 0) = 60,\ 221,\ 449,\ 1277,\ 3294 \quad \textit{for } c = 16, 24, 32, 40, 48,$$

every one of them as printed in [1, Table 4]*; in particular there are exactly* 1277 *semisimple balanced root systems of central charge* 40 *and* 449 *of central charge* 32*.*

(3) *At $c = 40$ exactly* 139 *simple types satisfy the bound of (1); the largest dual Coxeter number among them is* 78*, that of D_{40}, so $d_1 \le 40 \cdot 79 = 3160$ for every semisimple balanced root system of central charge* 40*.*

Proof sketch. (1) $k_i \ge 1$ in (1) gives $d_1 - c \le c\, h_i^\vee$, and $\dim \mathfrak{g}_i \le d_1$. (2) By (1) only finitely many simple types can occur at a given c (for A_n the bound reads $n \le c$, for D_n it reads $n \le c$, and so on), and by (1) again $d_1 \le c(1 + \max h^\vee)$ over those types. For each d_1 in that range the admissible types are those with $\dim \mathfrak{g} \le d_1$ and $(d_1 - c) \mid c\, h^\vee$, and the systems with that d_1 are the multiplicity vectors over the admissible types with $\sum_i \mathrm{mult}_i \dim \mathfrak{g}_i = d_1$. The machine-checked development contains an enumerator of such multiplicity vectors together with the theorem that it lists exactly the vectors with the prescribed weighted sum; the fifteen counts are then obtained by compiled evaluation of the resulting finite search. (3) is a further evaluation of the same data; the 139 types are $A_1, \ldots, A_{40}$, $B_2, \ldots, B_{39}$, $C_3, \ldots, C_{21}$, $D_4, \ldots, D_{40}$ and the five exceptional types. ☐

Proposition 6.2 (A correction to [1, Table 4])**.** *The number of semisimple balanced root systems of central charge* 10 *is $\#\,\mathrm{BRS}(10, 0) = 22$, not* 42 *as printed. The neighbouring entries $\#\,\mathrm{BRS}(9, 0) = 20$ and $\#\,\mathrm{BRS}(16, 0) = 60$ are reproduced, so the discrepancy is not a systematic offset of the enumeration.*

Proof. The same evaluation as in Theorem 6.1(2) at $c = 10$, 9 and 16. ☐

Remark 6.3 (Outside the formal development). The machine-checked statement is the count. The 22 systems returned by the enumeration, each with d_1 in parentheses, are $A_{2,30}A_{1,20}$ (11), $A_{1,10}^4$ (12), $G_{2,10}$ (14), $A_{1,4}^5$ (15), $A_{3,8}$ (15), $A_{2,5}^2$ (16), $A_{2,3}A_{1,2}^4$ (20), $B_{2,3}^2$ (20), $G_{2,4}A_{1,2}^2$ (20), $A_{1,1}^{10}$ (30), $A_{3,2}A_{1,1}^5$ (30), $A_{3,2}^2$ (30), $C_{3,2}A_{1,1}^3$ (30), $A_{2,1}^5$ (40), $B_{2,1}^4$ (40), $C_{3,1}A_{3,1}G_{2,1}$ (50), $B_{5,2}$ (55), $C_{4,1}A_{4,1}$ (60), $A_{5,1}^2$ (70), $D_{5,1}^2$ (90), $A_{10,1}$ (120) and $D_{10,1}$ (190); each was checked, outside the formal development, to satisfy $\sum_i k_i \dim \mathfrak{g}_i/(k_i + h_i^\vee) = 10$ exactly. The list itself is a by-product of that enumeration and is no part of Proposition 6.2, whose content is the count alone. Nothing in [1] depends on the entry.

7. Three cells at $c = 40$

The three rows of [2] treated in this section are rows 137 ($A_{6,35}$, $d_1 = 48$), 1221 ($A_{3,1}^4 G_{2,1}^{10}$, $d_1 = 200$) and 1276 ($A_{40,1}$, $d_1 = 1680$, the largest single-factor row of the table); none of them carries a recorded result, and their values of m are those of Proposition 2.5. They exhibit three different obstructions.

7.1. $A_{6,35}$**: a size obstruction.** Here $\mathfrak{g} = \mathfrak{sl}_7$, $h^\vee = 7$, $k = 35$, $\kappa = 42$, and by Lemma 2.6 conformal weight 2 means $\langle \lambda, \lambda + 2\rho \rangle = 168$, that is $c_6(\lambda) = 1176$ — the same Lie algebra as the $c = 32$ control $A_{6,14}$, at a different level.

Theorem 7.1 ($A_{6,35}$ is excluded)**.** *Let $\lambda = a\omega_1 + \cdots + f\omega_6$ be a dominant integral weight of $\mathfrak{sl}_7$ with $\langle \lambda, \lambda + 2\rho \rangle = 168$. Then $\dim V_\lambda \geq 194\,480$, $10 \mid \dim V_\lambda$, and $\operatorname{lev}(\lambda) = a + \cdots + f \leq 13$; the bound is attained at $\lambda = \omega_3 + 10\omega_6$ and at $10\omega_1 + \omega_4$, and there are exactly 56 such λ. Consequently, with D the set of top-level dimensions of the conformal-weight-two modules of $A_6^{(1)}$ at level 35, the number $m = 43204$ is not D-representable — both because $0 < m < \min D$ and because $\gcd(D)$ is divisible by 10 while $10 \nmid 43204$ — and the dimension test rules out the balanced root system $A_{6,35}$ at central charge 40.*

Proof sketch. The diagonal part (10) of c_6 gives $a \leq 10$ (since $6 \cdot 11^2 + 42 \cdot 11 = 1188 > 1176$), $b \leq 7$, $c, d \leq 7$ (here $12 \cdot 7^2 + 84 \cdot 7 = 1176$, so the bound is tight), $e \leq 7$, $f \leq 10$: a box $[0, 10] \times [0, 7]^4 \times [0, 10]$ of $495\,616$ tuples, against $\binom{41}{6} = 4\,496\,388$ dominant weights of level at most 35. An exhaustive evaluation over the box certifies, for every tuple with $c_6 = 1176$, the exactness of the Weyl quotient (11), the bound 194 480, the divisibility by 10 and the level bound 13; a second evaluation counts the solutions. Attainment is two evaluations, and $0 < 43204 < 194\,480$ then rules out every non-empty list of parts while the empty list sums to 0; alternatively 10 divides every part and not $43204 = 10 \cdot 4320 + 4$. □

7.2. $A_{3,1}^4 G_{2,1}^{10}$**: a coin-problem obstruction.** All fourteen factors are at level 1, and the level-one integrable weights are few: for $A_3^{(1)}$ they are $0, \omega_1, \omega_2, \omega_3$, of conformal weights $0, 3/8, 1/2, 3/8$ and top-level dimensions $1, 4, 6, 4$; for $G_2^{(1)}$ they are 0 and ω_1, of conformal weights 0 and $2/5$ and dimensions 1 and 7. In units of $1/40$ a tensor product has conformal weight $2 = 80/40$ exactly when

$$15p + 20q + 16r = 80, \qquad p + q \leq 4, \quad r \leq 10, \tag{13}$$

where p (resp. q) is the number of A_3 factors carrying ω_1 or ω_3 (resp. ω_2) and r the number of G_2 factors carrying ω_1; its top-level dimension is $4^p 6^q 7^r$.

Theorem 7.2 ($A_{3,1}^4 G_{2,1}^{10}$ is excluded)**.** *The set D of top-level dimensions of the conformal-weight-two modules of the affine algebra attached to $A_{3,1}^4 G_{2,1}^{10}$ is exactly $\{1296, 16807\}$, and $m = 100626$ is not D-representable: $1296a + 16807b = 100626$ has no solution in non-negative integers. Neither criterion stated in* [1, §5.1] *decides this row: $\gcd(D) = 1$, so $\gcd(D) \nmid m$ cannot apply; m lies below the Frobenius number $1296 \cdot 16807 - 1296 - 16807 = 21\,763\,769$ of $\{1296, 16807\}$, so the Frobenius criterion cannot apply either; and both parts are smaller than m, so there is no size obstruction. The dimension test rules out the balanced root system $A_{3,1}^4 G_{2,1}^{10}$ at central charge 40.*

Proof. The only solutions of (13) are $(p, q, r) = (0, 4, 0)$ and $(0, 0, 5)$, giving $6^4 = 1296$ and $7^5 = 16807$. Any sum of elements of D is $1296a + 16807b$; for it to equal 100626 one needs $b \leq 5$, and each of the six cases fails a congruence or a size check. The Frobenius number of two coprime parts x, y is $xy - x - y$. □

7.3. $A_{40,1}$**: no conformal-weight-two module at all.** At level 1 the integrable weights of $A_{40}^{(1)}$ are 0 and the fundamental weights ω_j, $1 \leq j \leq 40$, of conformal weight $j(41 - j)/82$ and top level $\Lambda^j \mathbb{C}^{41}$, of dimension $\binom{41}{j}$.

Theorem 7.3 ($A_{40,1}$ is excluded)**.** *No integrable module of $A_{40}^{(1)}$ at level 1 has conformal weight 2: the equation $j(41 - j) = 164$ has no solution with $0 \leq j < 41$. Hence $D = \varnothing$ while $m = 179820 > 0$, and the dimension test rules out the balanced root system $A_{40,1}$ at central charge 40.*

Proof. $j(41-j)$ takes the value 148 at $j=4$ and 180 at $j=5$, and is increasing on $0 \le j \le 20$ and symmetric about $j = 41/2$; a finite check of the 41 values settles it. Only the empty list of parts is available, and it sums to $0 \neq m$. □

Proposition 7.4 (Controls for the three cells)**.** *The following hold.*

(1) *The bound of Theorem 7.1 is sharp:* 194 481 *is not a lower bound, and* 194 480 is *D-representable for the D of that theorem.*
(2) *The Casimir condition selects: $\omega_3+9\omega_6$ and $\omega_3+11\omega_6$ do not have $c_6 = 1176$, and the minimiser $\omega_3+10\omega_6$ does not satisfy the $c = 32$ equation $c_6 = 588$ of Theorem 5.1.*
(3) *$A_{40}^{(1)}$ at level* 1 *does have modules —* $\binom{41}{4} = 101270$ *and* $\binom{41}{5} = 749398$ *are the dimensions of the two whose conformal weights bracket* 2*, since* $4 \cdot 37 = 148 < 164 < 180 = 5 \cdot 36$.
(4) *For $A_{3,1}^{4} G_{2,1}^{10}$: neither part alone divides* 100626*; the nearby target* $90515 = 5 \cdot 1296 + 5 \cdot 16807$ is *representable; and using all four A_3 factors at ω_1, or four G_2 factors, or three A_3 factors at ω_2 with one G_2 factor, misses conformal weight* 2.

Proof. Finite evaluations, using the witnesses of Theorems 7.1–7.3. □

8. Ninety more exclusions at $c = 40$

The three cells of §7 were done by hand, each with its own Casimir polynomial. The remaining exclusions rest on a generic engine, whose statement about modules and whose completeness theorem we now describe; the engineering that makes it run is described afterwards, and the ninety exclusions follow.

8.1. **The test as a statement about modules.** Let Φ be a semisimple balanced root system of central charge 40, written $\Phi = \mathfrak{g}_{1,k_1} \cdots \mathfrak{g}_{r,k_r}$ with its factors listed with multiplicity, and let $m = m(\Phi)$ be its value from (8). An irreducible U-module is a list $(\lambda_1, \ldots, \lambda_r)$ of dominant integral weights, λ_i of level at most k_i; its conformal weight is $\sum_i h(\lambda_i)$ with h as in (2), and its top-level dimension is $\prod_i \dim V_{\lambda_i}$.

Definition 8.1. We say that m is *realised by modules of* Φ if there is a finite list $M_1, \ldots, M_s$ ($s = 0$ allowed, repetitions allowed) of irreducible U-modules of conformal weight 2 with $\sum_j \dim(M_j)_{\mathrm{top}} = m$.

This is D-representability of m (Definition 2.1) with D left implicit, and it is the hypothesis the dimension test refutes. In the formal development it is the predicate `RepByModules`, quoted verbatim in §9: a module is there a list of label lists, one per factor, each of the right length, of level at most k_i, of conformal weight at most 2 and of positive computed Weyl dimension, with the conformal weights summing to exactly 2; every genuine module satisfies these side conditions, so refuting the formal predicate refutes Definition 8.1.

Proposition 8.2 (The engine)**.** *Let Φ and m be as above and let $D_{\le m}$ be the finite list of dimensions produced by the following procedure: for each factor, enumerate the dominant weights either of level at most k_i or inside the box that $\langle \lambda_i, \lambda_i + 2\rho_i \rangle \le 4\kappa_i$ forces, keep those satisfying both conditions, and record the pairs (conformal weight, $\dim V_{\lambda_i}$); combine the factors one at a time, keeping a partial tensor product only while its conformal weight is at most* 2 *and its dimension is at most m, optionally merging equal pairs; finally keep the dimensions of the products of conformal weight exactly* 2*. Then:*

(1) *(Completeness) every irreducible U-module of conformal weight* 2 *with top-level dimension at most m has its top-level dimension in $D_{\le m}$;*
(2) *(The prune is lossless) a part larger than m cannot occur in a representation of m, so if m is not $D_{\le m}$-representable then m is not realised by modules of Φ;*
(3) *(Three routes) m is not $D_{\le m}$-representable if $D_{\le m} = \varnothing$ and $m > 0$ (*size*); or if some g divides every element of $D_{\le m}$ but not m (*gcd*); or if a set $C \subseteq [0, m]$ containing* 0*, closed under adding an element of $D_{\le m}$ whenever the sum stays at most m, and not containing m is exhibited (*coin*).*

Proof sketch. (1) is an induction over the factors: the level enumeration is complete by definition and the box enumeration by Lemma 2.6 (in its generic form: the diagonal Gram entries of every type A to G are positive in the normalisation $\langle\theta,\theta\rangle = 2$, so each label is bounded by its own diagonal term), the filters keep exactly the weights that matter, a partial product of a module of dimension at most m has dimension at most m because every factor dimension is at least 1, and merging equal pairs preserves membership. (2) a part exceeds a sum of non-negative parts only if it is not among them. (3) the empty list sums to 0; a common divisor of the parts divides their sum; and C contains every sum of parts that is at most m, by induction on the list. □

In the development the three parts of Proposition 8.2 are the theorems `mem_wt2DimsSmall` (completeness, quoted in §9), `not_representable_of_small` together with `not_repByModules_of` (the prune, and the passage from the computed list back to the predicate `RepByModules` of Definition 8.1), and `excl_size`, `excl_gcd`, `excl_coin` (the three routes); the enumerations $D_{\le m}$ are the definitions `facPairs`, `combineDS` and `wt2DimsSmall`.

The classical inputs are those of §2 together with the Lie data of the types A to G (Cartan matrices in Bourbaki numbering, the symmetrised form with $\langle\theta,\theta\rangle = 2$, positive roots by reflection closure, comarks for the level, and the Weyl dimension formula as a product over positive roots), all of which enter the formal development as computable definitions, and the fact that the Weyl quotient is an integer at every dominant weight. Three engineering facts, measured rather than proved, are what make the ninety rows affordable: the dimension prune takes the total combiner work over the 94 rows found by the unformalised sweep of Remark 8.15 from about 10^{11} elementary steps (an estimate; the unpruned run was abandoned) to a measured $1.35 \cdot 10^7$, with a largest intermediate state of $1.9 \cdot 10^6$ pairs; the level enumeration beats the box by 26 orders of magnitude at $A_{40,1}$ (41 weights against a box of 3 823 249 346 281 680 666 624 000 000 points) while the box beats the level enumeration by a factor 3 at $A_{6,35}$ (1 587 600 against 4 496 388), so the choice is made per factor; and merging equal pairs takes the state of $A_{1,1}^{35}A_{3,2}$ from $\sum_{j\le 8}\binom{35}{j} \approx 3.3 \cdot 10^7$ entries to nine, but is a cost on rows whose state does not collapse, so it too is a per-factor switch. Neither switch affects the statement: the two enumerations agree as sets once filtered, and merging preserves membership.

8.2. **The ninety rows.** Each theorem below names its rows by the row numbers printed in the first column of [2] (the $\dim A = 0$ table), none of which carries a recorded result there; for each row the compiled evaluation re-derives d_1, checks the balanced relation $40h_i^\vee = k_i(d_1 - 40)$ on every factor, re-derives m from (8), and establishes the obstruction. "Excluded" means, as always, that m is not realised by modules of the root system (Definition 8.1), hence that the balanced root system fails the dimension test of [1, §5.1]. Appendix A lists all ninety rows with m and the obstruction; the tallies are 61 by gcd, 8 by size, 21 by coin.

Theorem 8.3 ($d_1 = 41$)**.** *The dimension test excludes the six balanced root systems of central charge* 40 *with* $d_1 = 41$ *in rows* 2, 3, 4, 5, 7, 8 *of* [2]*:* $B_{2,120}^2B_{3,200}$, $B_{2,120}^2C_{3,160}$, $A_{1,80}B_{2,120}D_{4,240}$, $A_{1,80}A_{4,200}G_{2,160}$, $A_{1,80}^2C_{3,160}G_{2,160}$ *and* $A_{1,80}B_{2,120}G_{2,160}^2$*, all with* $m = 40054$*. Rows* 2 *and* 3 *fail by size, rows* 4, 5, 8 *by gcd with* $g = 35, 1050, 35$*, and row* 7 *by coin.*

Theorem 8.4 ($d_1 = 42$)**.** *The dimension test excludes the nine balanced root systems of central charge* 40 *with* $d_1 = 42$ *in rows* 26, 27, 29, 30, 31, 32, 33, 40, 45 *of* [2]*:* $B_{3,100}^2$, $B_{3,100}C_{3,80}$, $C_{3,80}^2$, $D_{4,120}G_{2,80}$, $G_{2,80}^3$, $A_{1,40}A_{3,80}A_{4,100}$, $A_{2,60}A_{4,100}B_{2,60}$, $A_{2,60}B_{2,60}^2G_{2,80}$ *and* $A_{2,60}^4B_{2,60}$*, all with* $m = 40507$*. Rows* 26, 27, 29, 30, 31 *fail by size and rows* 32, 33, 40, 45 *by coin.*

Theorem 8.5 ($d_1 = 43, 44$)**.** *The dimension test excludes the four balanced root systems of central charge* 40 *in rows* 57, 62, 68, 71 *of* [2]*:* $A_{2,40}A_{5,80}$ *(*$d_1 = 43$*,* $m = 40959$*, gcd with* $g = 22680$*),* $A_{4,50}B_{2,30}^2$ *(*$d_1 = 44$*,* $m = 41410$*, coin),* $A_{3,40}^2G_{2,40}$ *(*$d_1 = 44$*,* $m = 41410$*, gcd with* $g = 3$*) and* $B_{2,30}^3G_{2,40}$ *(*$d_1 = 44$*,* $m = 41410$*, coin).*

Theorem 8.6 ($d_1 = 48$)**.** *The dimension test excludes the eight balanced root systems of central charge* 40 *with* $d_1 = 48$ *in rows* 138, 139, 140, 148, 155, 158, 163, 164 *of* [2]*:* $A_{1,10}D_{5,40}$, $A_{4,25}^2$, $A_{1,10}A_{5,30}B_{2,15}$,

$B_{2,15}^2D_{4,30}$, $A_{1,10}A_{3,20}^3$, $A_{2,15}A_{3,20}^2B_{2,15}$, $A_{1,10}A_{3,20}B_{2,15}^3$ and $A_{2,15}B_{2,15}^4$, all with $m = 43204$. Row 139 fails by coin, and the other seven by gcd with $g = 8505, 35, 35, 5, 5, 5, 5$ respectively.

Theorem 8.7 ($d_1 = 60, 105$)**.** *The dimension test excludes the three balanced root systems of central charge 40 in rows 274, 280, 924 of [2]: $A_{3,8}A_{4,10}B_{3,10}$ and $A_{3,8}A_{4,10}C_{3,8}$ ($d_1 = 60$, $m = 48490$, both by coin) and $B_{7,8}$ ($d_1 = 105$, $m = 67030$, by size).*

Theorem 8.8 ($d_1 = 120$)**.** *The dimension test excludes the forty-two balanced root systems of central charge 40 with $d_1 = 120$ in rows 950, 951, 954, 955, 956, 958, 962, 963, 970, 971; 975, 976, 979, 981, 982, 984, 985, 988, 989, 990; 1003, 1006, 1018, 1019, 1025, 1026, 1028, 1039, 1040, 1066; and 1067, 1101, 1104, 1109, 1110, 1113, 1114, 1115, 1117, 1118, 1119, 1120 of [2] (four groups of ten, ten, ten and twelve), listed with their root systems in Appendix A. Their values of m lie between 72760 and 72935 and are determined by the level-one correction of (8): $m = 72760$ on the rows without a level-one factor, and $m = 72760+5j$ on the rows with exactly j factors $A_{1,1}$ (each contributing $\dim V_{2\theta}(\mathfrak{sl}_2) = 5$). Thirty-four rows fail by gcd, with $g \in \{2, 4, 7, 8, 14, 16, 35, 64\}$, and eight (951, 954, 956, 982, 1003, 1019, 1039, 1066) by coin.*

Theorem 8.9 ($d_1 = 160, 180$)**.** *The dimension test excludes the seven balanced root systems of central charge 40 in rows 1138, 1142, 1143, 1144, 1184, 1191, 1192 of [2]: $A_{2,1}^3A_{5,2}B_{2,1}D_{7,4}$ ($m = 87056$, gcd with $g = 7$), $A_{2,1}^5B_{2,1}B_{5,3}^2$ and $A_{2,1}^5B_{2,1}B_{5,3}C_{5,2}$ ($m = 87110$, gcd with $g = 11$), $A_{2,1}^5B_{2,1}C_{5,2}^2$ ($m = 87110$, coin), $A_{2,1}^5B_{2,1}^{12}$ ($m = 87495$, gcd with $g = 25$), all with $d_1 = 160$; and $B_{4,2}^5$ (coin) and $A_{6,2}^3B_{4,2}$ (gcd with $g = 4$), with $d_1 = 180$ and $m = 93430$.*

Theorem 8.10 ($d_1 = 200, 240$)**.** *The dimension test excludes the eleven balanced root systems of central charge 40 in rows 1207, 1208, 1209, 1210, 1211, 1214, 1215, 1218, 1222, 1223, 1241 of [2]: $A_{3,1}^3C_{3,1}^3E_{6,3}G_{2,1}$ ($m = 100227$, coin), $A_{3,1}A_{7,2}C_{3,1}D_{5,2}G_{2,1}^4$ (100038, $g = 7$), $A_{3,1}^3C_{3,1}E_{6,3}G_{2,1}^4$ (100206, $g = 7$), $A_{3,1}^4A_{7,2}^2G_{2,1}$ (99933, $g = 9$), $A_{3,1}C_{3,1}^6D_{5,2}G_{2,1}$ (100437, $g = 12$), $A_{3,1}D_{5,2}G_{2,1}^{10}$ (100374, $g = 343$), $A_{3,1}^4A_{7,2}C_{3,1}^3G_{2,1}$ (100311, $g = 12$), $A_{3,1}^4C_{3,1}^6G_{2,1}$ (100689, $g = 24$), $A_{3,1}^8C_{3,1}D_{5,2}G_{2,1}$ (100395, $g = 12$), $A_{3,1}^{11}C_{3,1}G_{2,1}$ (100647, $g = 24$), all with $d_1 = 200$; and $A_{4,1}^3B_{3,1}^8$ ($d_1 = 240$, $m = 112444$, $g = 49$). All but the first fail by gcd with the stated g.*

Proof of Theorems 8.3–8.10. Each row is one compiled evaluation of the engine of Proposition 8.2 on the factor list, with a common denominator S for the conformal weights chosen so that $2\kappa_i\langle\theta_i, \theta_i\rangle^{-1}\cdot\mathrm{den}_i$ divides S for every factor (den_i the denominator of the Gram matrix of $\mathfrak{g}_i$); the evaluation checks that the factor data is consistent, that $40h_i^\vee = k_i(d_1 - 40)$ on every factor, that d_1 and m have the stated values, and then either that $D_{\le m}$ is empty, or that g divides every entry of $D_{\le m}$ (and $g \nmid m$ is a separate check), or that every entry of $D_{\le m}$ lies in a stated list of parts and that the reachable set of $[0, m]$ under those parts, computed in one forward pass, is closed and misses m. Proposition 8.2 then converts each into "m is not realised by modules". The enumeration choices per factor are recorded in the factor lists. □

Remark 8.11. The coin obstruction is the interesting one: at these twenty-one rows the divisibility criterion of [1, §5.1] is reported not to apply — the engine tries the gcd route first, so a common divisor of the parts not dividing m would have been found there — and the unformalised sweep of Remark 8.15 reports $\gcd(D_{\le m}) = 1$ at eleven of them (rows 32, 33, 40, 45, 71, 274, 280, 954, 982, 1003, 1039) and m below the Frobenius number at all of them. Both are reports of the computation, not machine-checked statements: what the theorems assert at these rows is non-representability alone, and what establishes it is the reachability computation itself. At row 274 the value gcd = 1 is machine-checked and the obstruction is one unit wide (Proposition 8.14(3)).

8.3. **A row the ancillary table marks as excluded, and which is not.** Row 93 of [2] is $A_{5,48}B_{2,24}$, $d_1 = 45$, $m = 495 \cdot 45 + 20620 - 1035 = 41\,860$ (no level-one factor). The printed entry of that row, as extracted from the PDF, is

```
93  45  A5,48 B2,24  X (d)
```

that is, row number 93, $\dim V_1 = 45$, root system $A_{5,48}B_{2,24}$, result "X (d)".

Theorem 8.12 (Row 93 is not excluded)**.** *The number $m = 41\,860$ is realised by modules of $A_{5,48}B_{2,24}$: the modules $L(48, 0) \otimes L(24, 9\omega_1)$ and $L(48, 0) \otimes L(24, 4\omega_1 + 8\omega_2)$ of the affine algebra attached to $A_{5,48}B_{2,24}$ have conformal weight 2 and top-level dimensions 385 and 1995 (levels 9 and 12 on the B_2 factor), and*

$$41\,860 = 88 \cdot 385 + 4 \cdot 1995.$$

Hence the dimension test does not exclude the balanced root system $A_{5,48}B_{2,24}$ at central charge 40, contrary to the entry "X (d)" of [2]. Moreover 385 and 1995 are the only top-level dimensions at most m of conformal-weight-two modules of this root system.

Proof sketch. On the B_2 factor $\kappa = 27$, so conformal weight 2 with the A_5 factor in its vacuum means $\langle\lambda, \lambda+2\rho\rangle = 108$. In Bourbaki numbering (ω_1 the 5-dimensional and ω_2 the 4-dimensional representation of $\mathfrak{so}_5$) the normalisation $\langle\theta, \theta\rangle = 2$ gives $\langle\omega_1, \omega_1\rangle = 1$ and $\langle\omega_1, \omega_2\rangle = \langle\omega_2, \omega_2\rangle = \frac{1}{2}$, whence $2\langle\lambda, \lambda+2\rho\rangle = 2p^2+2pq+q^2+6p+4q$; the Weyl formula reads $\dim V_{p\omega_1+q\omega_2} = (p+1)(q+1)(2p+q+3)(p+q+2)/6$, and the level is $\mathrm{lev}(\lambda) = p+q$, both comarks of B_2 being 1. The equation $2p^2+2pq+q^2+6p+4q = 216$ has exactly the solutions $(9, 0)$ and $(4, 8)$ with $p + q \leq 24$, of levels 9 and 12 and dimensions 385 and 1995. The formal development does not use these closed forms: it checks the two modules through the generic Lie data (the Casimir value, the level and the Weyl dimension of each), it exhibits the multiset of $88+4$ modules, and it establishes the last sentence by the engine of Proposition 8.2, whose $D_{\leq m}$ for this row consists of the two values 385 and 1995. The closed forms are recorded only as an aid to the reader; what is machine-checked is the Casimir value $108 = 4\kappa$, the two levels and the two dimensions. ☐

Remark 8.13. Theorem 8.12 is a non-exclusion, so, as in Theorem 4.1, it is a witness and is insensitive to any incompleteness in D; it asserts nothing about the existence of a vertex operator algebra. It is also this paper's positive control at $c = 40$: an engine that excluded every row would prove nothing. It is one of the 24 rows on which the unformalised sweep of Remark 8.15 disagrees with the "X (d)" entries of [2]; we assert no cause for the disagreement — the authors' code is not published — only the disagreement, now with a witness.

Proposition 8.14 (Controls for the ninety rows)**.** *The following hold.*

(1) *The size obstruction is about size, not emptiness: $B_{7,8}$ (row 924) has conformal-weight-two modules at $\omega_1 + \omega_2 + \omega_3$, $2\omega_1 + \omega_5$ and $\omega_2 + \omega_6$, of dimensions 230 945, 271 700 and 388 960, all above $m = 67\,030$; and $B^2_{3,100}$ (row 26) has one at $(\omega_1, 18\omega_1)$, of dimension 419 881, above $m = 40\,507$.*

(2) *A too-small claim is refuted: "every conformal-weight-two module of $B_{7,8}$ has dimension at least 271 700" is false, 230 945 being attained.*

(3) *The coin obstruction at row 274 is sharp: the parts have $\gcd = 1$, so the divisibility criterion cannot decide the row, and $m - 1 = 48\,489 = 59 \cdot 735 + 3 \cdot 924 + 2 \cdot 1176$ is representable while $m = 48\,490$ is not.*

(4) *The level-one correction bites: at row 1223 ($A^{11}_{3,1}C_{3,1}G_{2,1}$) it equals 1127 and moves m from 99 520 to 100 647.*

(5) *The conformal-weight condition bites: perturbing the B_2 label 9 of Theorem 8.12 to 8 or 10 misses conformal weight 2.*

(6) *Engine controls: merging equal pairs really removes something; the level and box enumerations give the same pair list at A_3 level 1, G_2 level 1, C_3 level 2 and B_2 level 1; their sizes differ as stated after Proposition 8.2; and the engine reproduces $d_1 = 200$, the correction 1106, $m = 100626$ and $D_{\leq m} = \{1296, 16807\}$ of Theorem 7.2 through an independent code path.*

Proof. Finite evaluations with explicit witnesses, through the same engine. ☐

Remark 8.15 (The whole $c = 40$ table; outside the formal development)*.* The same re-implementation as in Remark 5.4, extended by the prune of Proposition 8.2(2), was run on all 1277 rows of [2], whose root systems were parsed from the PDF by superscript and subscript position and validated by $\sum_i \mathrm{mult}_i \dim \mathfrak{g}_i = d_1$ and $40h_i^\vee = k_i(d_1 - 40)$ on every row, and independently reproduced, set for set, by the enumerator of §6. Of the 1123 rows with no recorded result, the sweep excludes 94: the 93 of Theorems 7.1–8.10 and row 1, $A_{1,80}^2 A_{5,240}$ ($m = 40054$, every part at most m divisible by 35), which is not carried into the formal development because the box of its $A_{5,240}$ factor has 21 068 100 points, beyond what the present engine materialises within its memory budget. Of the 118 rows labelled "X (d)", the sweep reproduces 94 and finds 24 representable (rows 93, 95, 99, 101, 102, 265, 395, 404, 411, 413, 416, 907, 911, 912, 934, 1125, 1130, 1131, 1134, 1200, 1203, 1227, 1230, 1250), every one of which has at least two simple factors; all 36 rows labelled "X (J)" pass the dimension test, as that label requires. The one tempting explanation of the 24 — that the source's D omits tensor products with a trivial factor — accounts for 117 of the 118 labels but is refuted by the source's own $c = 32$ table, under which it would exclude all 36 remaining "?" rows and 8 of the 11 realised lattice rows; so no cause is asserted. The 94 exclusions and the 24 disagreements are disjoint sets of rows. A second, independent implementation mirroring the formal development's Lie data function for function re-derived d_1, m, the obstruction class and $\gcd(D_{\leq m})$ on all 94 rows with no disagreement. The remaining 1029 blank rows pass the dimension test, and stay open for the other two tests, which we have not attempted.

9. Verification

Every numbered statement above is machine-checked by the Lean 4 proof assistant [7]. The classical inputs of §2 — the graded dimensions (4), (5), the grade-two vacuum count (6), the conformal-weight formula (2), the level-one integrable weights of $A_n^{(1)}$ and $G_2^{(1)}$ used in §7, and the Weyl dimension formula (11) — enter as definitions rather than as formalised theory; what is machine-checked is therefore the finite mathematics these reduce to, not the vertex-algebra background. The $c = 32$ development and the census and the three cells of §7 are checked over the Mathlib library [8]; the engine of §8 and the ninety rows are checked over a self-contained Lie-data layer without Mathlib (Cartan matrices, symmetrisers, positive roots by reflection closure, comarks, Casimir values, Weyl dimensions and the two enumerations, each with its membership lemma), which is what keeps the memory footprint of the ninety rows below two gigabytes. The development contains no `sorry` and declares no axiom of its own. The formal statements are the ones printed above, with the following provisos. In Theorem 4.1 the set the formal statement quantifies over is the subset of D given by the conformal-weight-two modules supported on the three $A_{1,64}$ factors alone; representability over that subset implies the stated representability over D. In Theorems 7.2 and 7.3 the set D is formalised through the parametrisations (13) and $j \mapsto \binom{41}{j}$, the passage from modules to those parametrisations being the classical input named in §7. In §8 the Weyl dimension is a computed integer quotient, and its agreement with $\dim V_\lambda$ at dominant weights is classical input (in the two $c = 32$ sweeps and the $A_{6,35}$ sweep the exactness of the quotient is certified inside the search). The remarks are outside the formal development, as each says of itself.

The dimension test at $c = 40$ is stated about modules through the following definition, quoted from the development with its symbols typeset and its lines re-broken (`FacM` is a factor with its two enumeration switches, type, level and multiplicity; `expandF` expands the multiplicities; `choiceOKF` is the legality of a list of label lists, `choicePosF` the positivity of its Weyl dimensions, `confTotalF` and `dimTotalF` its scaled conformal weight and its top-level dimension):

```
def RepByModules (fs : List FacM) (S m : Nat) : Prop :=
    ∃ chs : List (List (List Nat)),
      (∀ ls ∈ chs, choiceOKF (expandF fs) ls ∧ choicePosF (expandF fs) ls ∧
        confTotalF S (expandF fs) ls = 2 * S) ∧
      sumN (chs.map (dimTotalF (expandF fs))) = m
```

and Proposition 8.2(1) is the theorem

```
theorem mem_wt2DimsSmall (S m : Nat) (fs : List FacM) (ls : List (List Nat))
      (hok : choiceOKF (expandF fs) ls) (hpos : choicePosF (expandF fs) ls)
      (heq : confTotalF S (expandF fs) ls = 2 * S)
      (hdim : dimTotalF (expandF fs) ls ≤ m) :
      dimTotalF (expandF fs) ls ∈ wt2DimsSmall fs S m
```

Each of the ninety rows is a definition and a theorem of the shape

```
def r2 : List FacM := [(false, false, .B 2, 120, 2), (false, false, .B 3, 200, 1)]
   theorem r2_excluded : ¬ RepByModules r2 4920 40054
```

(this is row 2; the ninety statements are listed in Appendix A, each with its denominator S), and Theorem 8.12 is

```
def r93 : List FacM := [(false, false, .B 2, 24, 1), (false, false, .A 5, 48, 1)]
  theorem r93_not_excluded : RepByModules r93 648 41860
```

The two Boolean entries of a factor are not consulted by `RepByModules`, which reads only the type, the level and the multiplicity; they steer the computation alone.

Axioms. The axioms of every theorem were listed by the proof assistant's own `#print axioms` on 4 September 2026. The proof layer of the engine (Proposition 8.2, twenty-two declarations) depends on `propext` and, in part, on `Classical.choice` and `Quot.sound`, and on nothing else; so do Theorems 4.1 and 7.2 (and Theorem 7.3 on `propext` alone), and the finiteness and enumerator theorems behind Theorem 6.1(1)–(2). Every exhaustive search is run by compiled evaluation, which contributes one named compiled-evaluation axiom per search: one apiece for Theorems 3.1, 3.3, 5.1, 7.1 (two, the second for the count 56), each of the fifteen counts of Theorem 6.1(2) and Proposition 6.2, and each of the ninety row theorems of Theorems 8.3–8.10 (whose axioms are exactly `propext`, `Classical.choice`, `Quot.sound` and that one axiom); Theorem 8.12 uses compiled evaluation for the finite data of its two witnesses and of the multiset, and the controls of Propositions 5.3, 7.4 and 8.14 use it where they evaluate. *Cost.* The figures that follow are as recorded in the check logs of 3 September 2026, when the modules were checked; they were not re-measured for the present version, and they are of course machine-dependent. The $c = 32$ modules took about twenty seconds of processor time for the search module and half that for the rest, peaking 0.6 gigabytes above an `import Mathlib` baseline of 6.6 gigabytes; the census and the three $c = 40$ cells about 44 seconds together at a similar footprint; and the engine, the eleven row modules and the controls 645 seconds of processor time in all (10.8 minutes, of which the three modules holding rows 2–71 account for 510), peaking at 1.92 gigabytes.

The source of the development is currently held in a private repository: repository reference to be added at submission.

On the reference list. Reference [1] was read here from its arXiv LaTeX e-print of v3, every passage quoted above was re-read at its own line in that source, and the e-print was compiled so that the internal numbers quoted (Definitions 2 and 10, Table 4, Lemma 5, Theorems 1 and 12, §§5.1 and 5.4, and the row numbers of Appendix A) are the ones v3 prints. Its author list, title and version history were re-checked on its arXiv abstract page on 4 September 2026, where v3 of 20 March 2026 is current (v2 having been withdrawn) and no journal reference is recorded; all numbering above is therefore pinned to v3 and could move in a later version. The ancillary file [2] is the one attached to v3 (its listed size, 403 537 bytes, agrees with the copy read here); it carries the date 12 February 2026 and no version number of its own, and its row numbers are those printed in its first column. References [3], [4], [5] and [6] are cited for background and attribution, and none of their contents is relied on for a claim above beyond the standard affine and Weyl formulas (2) and (11).

10. Changes since the first version

The first version of this paper (3 September 2026) contained §§1–5 restricted to $c = 32$. This version keeps those sections as refereed, with three additions to §§1–2: the balanced relation and the formula for m are now stated for $c \in \{32, 40\}$, with (5), (8) and Proposition 2.5; the introduction quotes the source on $c = 40$ and describes the ancillary table; and the results at $c = 40$ are summarised. New are §6 (the census and the correction at $c = 10$), §7 (three cells), §8 (the engine, the ninety exclusions, the correction at row 93, the controls and the sweep remark), Appendix A, and the corresponding parts of §9. The title has changed accordingly. No statement of the first version has been withdrawn or weakened.

Appendix A. The ninety rows

The table lists, for each of the ninety rows of Theorems 8.3–8.10, its row number ρ in [2], the root system (multiplicities as exponents, as in [2]), d_1, m from (8), the obstruction ("gcd g" for the gcd route with divisor g), and the common denominator S of the formal statement, which is the theorem `rρ_excluded` with statement ¬ `RepByModules` `rρ` S m; rows 2–924 are in the left half and rows 950–1241 in the right half.

row	root system	d_1	m	obstr.	S	row	root system	d_1	m	obstr.	S
2	$B_{2,120}^2B_{3,200}$	41	40054	size	4920	984	$A_{3,2}C_{3,2}^3G_{2,2}^3$	120	72760	gcd 7	144
3	$B_{2,120}^2C_{3,160}$	41	40054	size	1968	985	$A_{3,2}A_{5,3}D_{4,3}G_{2,2}^3$	120	72760	gcd 7	432
4	$A_{1,80}B_{2,120}D_{4,240}$	41	40054	gcd 35	1968	988	$A_{3,2}C_{3,2}D_{4,3}G_{2,2}^4$	120	72760	gcd 7	144
5	$A_{1,80}A_{4,200}G_{2,160}$	41	40054	gcd 1050	24600	989	$A_{3,2}A_{5,3}G_{2,2}^5$	120	72760	gcd 7	432
7	$A_{1,80}^2C_{3,160}G_{2,160}$	41	40054	coin	1968	990	$A_{3,2}C_{3,2}G_{2,2}^6$	120	72760	gcd 7	144
8	$A_{1,80}B_{2,120}G_{2,160}^2$	41	40054	gcd 35	984	1003	$A_{1,1}^{11}C_{3,2}D_{6,5}$	120	72815	coin	120
26	$B_{3,100}^2$	42	40507	size	840	1006	$A_{1,1}^5C_{3,2}D_{4,3}^3$	120	72785	gcd 14	72
27	$B_{3,100}C_{3,80}$	42	40507	size	1680	1018	$A_{1,1}^5C_{3,2}D_{4,3}^2G_{2,2}^2$	120	72785	gcd 7	72
29	$C_{3,80}^2$	42	40507	size	336	1019	$A_{1,1}^{14}E_{6,6}$	120	72830	coin	108
30	$D_{4,120}G_{2,80}$	42	40507	size	1008	1025	$A_{1,1}^5C_{3,2}D_{4,3}G_{2,2}^4$	120	72785	gcd 7	72
31	$G_{2,80}^3$	42	40507	size	504	1026	$A_{1,1}^5A_{5,3}G_{2,2}^5$	120	72785	gcd 7	108
32	$A_{1,40}A_{3,80}A_{4,100}$	42	40507	coin	16800	1028	$A_{1,1}^5C_{3,2}G_{2,2}^6$	120	72785	gcd 7	72
33	$A_{2,60}A_{4,100}B_{2,60}$	42	40507	coin	37800	1039	$A_{1,1}^{13}A_{3,2}D_{6,5}$	120	72825	coin	240
40	$A_{2,60}B_{2,60}^2G_{2,80}$	42	40507	coin	1512	1040	$A_{1,1}^7A_{3,2}D_{4,3}^3$	120	72795	gcd 2	144
45	$A_{2,60}^4B_{2,60}$	42	40507	coin	1512	1066	$A_{1,1}^{18}D_{6,5}$	120	72850	coin	120
57	$A_{2,40}A_{5,80}$	43	40959	gcd 22680	1032	1067	$A_{1,1}^{12}D_{4,3}^3$	120	72820	gcd 8	72
62	$A_{4,50}B_{2,30}^2$	44	41410	coin	6600	1101	$A_{1,1}^{19}C_{3,2}D_{4,3}G_{2,2}$	120	72855	gcd 2	72
68	$A_{3,40}^2G_{2,40}$	44	41410	gcd 3	1056	1104	$A_{1,1}^{19}C_{3,2}G_{2,2}^3$	120	72855	gcd 2	72
71	$B_{2,30}^3G_{2,40}$	44	41410	coin	264	1109	$A_{1,1}^{21}A_{3,2}D_{4,3}G_{2,2}$	120	72865	gcd 2	144
138	$A_{1,10}D_{5,40}$	48	43204	gcd 8505	384	1110	$A_{1,1}^{21}A_{3,2}G_{2,2}^3$	120	72865	gcd 2	144
139	$A_{4,25}^2$	48	43204	coin	300	1113	$A_{1,1}^{26}C_{3,2}^2$	120	72890	gcd 4	24
140	$A_{1,10}A_{5,30}B_{2,15}$	48	43204	gcd 35	432	1114	$A_{1,1}^{26}D_{4,3}G_{2,2}$	120	72890	gcd 4	72
148	$B_{2,15}^2D_{4,30}$	48	43204	gcd 35	288	1115	$A_{1,1}^{26}G_{2,2}^3$	120	72890	gcd 8	36
155	$A_{1,10}A_{3,20}^3$	48	43204	gcd 5	192	1117	$A_{1,1}^{28}A_{3,2}C_{3,2}$	120	72900	gcd 8	48
158	$A_{2,15}A_{3,20}^2B_{2,15}$	48	43204	gcd 5	1728	1118	$A_{1,1}^{30}A_{3,2}^2$	120	72910	gcd 16	48
163	$A_{1,10}A_{3,20}B_{2,15}^3$	48	43204	gcd 5	576	1119	$A_{1,1}^{33}C_{3,2}$	120	72925	gcd 16	24
164	$A_{2,15}B_{2,15}^4$	48	43204	gcd 5	432	1120	$A_{1,1}^{35}A_{3,2}$	120	72935	gcd 64	48
274	$A_{3,8}A_{4,10}B_{3,10}$	60	48490	coin	2400	1138	$A_{2,1}^3A_{5,2}B_{2,1}D_{7,4}$	160	87056	gcd 7	384
280	$A_{3,8}A_{4,10}C_{3,8}$	60	48490	coin	2400	1142	$A_{2,1}^5B_{2,1}B_{5,3}^2$	160	87110	gcd 11	96
924	$B_{7,8}$	105	67030	size	168	1143	$A_{2,1}^5B_{2,1}B_{5,3}C_{5,2}$	160	87110	gcd 11	96
950	$A_{3,2}A_{7,4}D_{4,3}G_{2,2}$	120	72760	gcd 7	576	1144	$A_{2,1}^5B_{2,1}C_{5,2}^2$	160	87110	coin	96
951	$A_{1,1}^4D_{4,3}D_{6,5}G_{2,2}$	120	72780	coin	360	1184	$A_{2,1}^5B_{2,1}^{12}$	160	87495	gcd 25	96
954	$A_{1,1}^7C_{3,2}E_{6,6}$	120	72795	coin	216	1191	$B_{4,2}^5$	180	93430	coin	72
955	$A_{3,2}A_{7,4}G_{2,2}^3$	120	72760	gcd 7	576	1192	$A_{6,2}^3B_{4,2}$	180	93430	gcd 4	504
956	$A_{1,1}^4D_{6,5}G_{2,2}^3$	120	72780	coin	360	1207	$A_{3,1}^3C_{3,1}^3E_{6,3}G_{2,1}$	200	100227	coin	360
958	$A_{3,2}A_{5,3}^3$	120	72760	gcd 35	432	1208	$A_{3,1}A_{7,2}C_{3,1}D_{5,2}G_{2,1}^4$	200	100038	gcd 7	480
962	$A_{3,2}C_{3,2}^5$	120	72760	gcd 7	48	1209	$A_{3,1}^3C_{3,1}E_{6,3}G_{2,1}^4$	200	100206	gcd 7	360
963	$A_{3,2}A_{5,3}C_{3,2}^2D_{4,3}$	120	72760	gcd 7	432	1210	$A_{3,1}^4A_{7,2}^2G_{2,1}$	200	99933	gcd 9	480
970	$A_{3,2}C_{3,2}D_{4,3}^3$	120	72760	gcd 14	144	1211	$A_{3,1}C_{3,1}^6D_{5,2}G_{2,1}$	200	100437	gcd 12	240
971	$A_{3,2}A_{5,3}^2C_{3,2}G_{2,2}$	120	72760	gcd 7	432	1214	$A_{3,1}D_{5,2}G_{2,1}^{10}$	200	100374	gcd 343	240
975	$A_{3,2}C_{3,2}^3D_{4,3}G_{2,2}$	120	72760	gcd 7	144	1215	$A_{3,1}^4A_{7,2}C_{3,1}^3G_{2,1}$	200	100311	gcd 12	480
976	$A_{3,2}A_{5,3}D_{4,3}^2G_{2,2}$	120	72760	gcd 7	432	1218	$A_{3,1}^4C_{3,1}^6G_{2,1}$	200	100689	gcd 24	120
979	$A_{3,2}A_{5,3}C_{3,2}^2G_{2,2}^2$	120	72760	gcd 7	432	1222	$A_{3,1}^8C_{3,1}D_{5,2}G_{2,1}$	200	100395	gcd 12	240
981	$A_{3,2}C_{3,2}D_{4,3}^2G_{2,2}^2$	120	72760	gcd 7	144	1223	$A_{3,1}^{11}C_{3,1}G_{2,1}$	200	100647	gcd 24	120
982	$A_{1,1}^9A_{3,2}E_{6,6}$	120	72805	coin	432	1241	$A_{4,1}^3B_{3,1}^8$	240	112444	gcd 49	240

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