EVALUATING AN AVOIDANCE NUMBER: $A_3(4) = 9$, $A_3(5) \geq 18$, AND TWO BRACKETS FOR ORIENTED RAMSEY NUMBERS

KOREA SUPERINTELLIGENCE LABS

Abstract. Duman, Gönül, Kaya, Saxena and Tamer introduce two finite Ramsey-type quantities as the inputs to a constant in their work on closed Ramsey numbers of small countable ordinals: the directed Ramsey number $R(K_n^*, L_3)$ and an *avoidance number* $A_3(k)$, the least m such that every 3-colouring of the arcs of K_m^* admits a k-set S in which every vertex misses a colour on its arcs from S. They evaluate neither. We first observe that $R(K_m^*, L_3)$ is the oriented Ramsey number $r(I_m, L_3)$, for which exact values up to $m = 5$ and the bound $r(I_m, L_3) \leq m^2 - m + 3$ are in print but are not cited in that paper. We then determine $A_3(4) = 9$, the first value of the avoidance number beyond the trivial $A_3(3) = 3$, against a published bracket whose upper end is the unknown three-colour Ramsey number $R(4, 4, 4)$. The proof rests on a symmetry of A_3 that acts independently at each vertex; we also give the best upper bound we can prove with no search at all, $A_3(4) \leq 17$, from a new obstruction on *pairs* of vertices, and the first nontrivial lower bound at $k = 5$, $A_3(5) \geq 18$. For the directed function we prove two brackets for which we could find no published counterpart, $29 \leq r(I_6, L_3) \leq 33$ and $21 \leq r(I_3, L_4) \leq 25$; in each the upper end is Ihringer, Rajendraprasad and Weinert's published bound and the lower end is new, the second being the cell they name as the natural next question and where only $r(I_3, L_4) \geq 9$ was available. Every statement below has been machine-checked.

1. Introduction

1.1. Two functions defined and not evaluated. In their study of closed Ramsey numbers of small countable ordinals, Duman, Gönül, Kaya, Saxena and Tamer [4] introduce, in the section "Coloring complete directed graphs", two finite quantities. Write K_N^* for the complete directed graph on N vertices, in which every ordered pair of distinct vertices carries an arc.

The first is a directed Ramsey number. Colour the arcs of K_N^* red and blue. A *red* K_m^* is a set of m vertices both of whose arcs are red for every pair inside it; a *blue* L_3 is a triple a, b, d of distinct vertices with $a \to b$, $b \to d$ and $a \to d$ all blue — a blue transitive tournament, the reverse arcs being unconstrained. Then $R(K_m^*, L_3)$ is the least N such that every red/blue colouring of K_N^* contains one or the other. For this quantity [4] quotes Larson and Mitchell's bound $R(K_m^*, L_3) \leq m^2$ [9] and no value.

The second is what we shall call the *avoidance number*. Their Proposition 1 and the sentences following it read, verbatim:

> "Let $k \geq 3$ be an integer. Then there exists an integer $m \geq 1$ such that for every edge coloring of the directed complete graph K_m^* with 3 colors, there exists a subset $S \subseteq K_m^*$ of size k such that: For every $x \in S$, the edges from vertices in S to x avoid at least

> one color. … we shall denote the smallest such positive integer by $A_3(k)$. … we have $A_3(k) \le R(k,k,k)$. It is also not difficult to check that $A_3(k) \ge 3^{k-2}/2^{k-1}$. While it is likely that this concept or an equivalent concept may have been already studied in the existing literature, we were unable to determine if this is indeed the case."

The two are the finite inputs to the constant $M(n) = R(A_3(R(K_n^*, L_3)), 3)$ of that paper, which is why they are defined at all. At $k = 3$ the published bracket for A_3 reads $0.75 \le A_3(3) \le R(3,3,3) = 17$; at $k = 4$ it reads $1.125 \le A_3(4) \le R(4,4,4)$, and $R(4,4,4)$ is itself unknown — the published bounds are $128 \le R(4,4,4) \le 229$, the lower one recorded in [12, Table XIa] and the upper one Boza's [2], which improves by one the value 230 that the general inequality [12, 6.1.a] gives.

1.2. **The first observation: one of the two is published under another name.** $R(K_m^*, L_3)$ is the oriented Ramsey number written $r(I_m, L_3)$ in the literature: the least N such that every oriented graph on N vertices contains an independent set of size m or a transitive tournament on three vertices. The two formulations agree (Lemma 2.3 below). Under that name the quantity has been studied for fifty years: $r(I_2, L_3) = 4$ goes back to Erdős and Rado [6], $r(I_3, L_3) = 9$ to Bermond [1], and Ihringer, Rajendraprasad and Weinert [8] prove $r(I_4, L_3) = 15$ and $r(I_5, L_3) = 23$ together with

$$r(I_m, L_3) \ \le\ m^2 - m + 3 \qquad (m \ge 3),$$

which improves Larson and Mitchell's m^2 and is tight for $m \in \{3,4,5\}$. The bridge between the two literatures is the ordinal identity $R(\omega \cdot m, k) = \omega \cdot r(I_m, L_k)$ of Erdős and Rado, recorded by Caicedo and Hilton [3].

Neither [8] nor [1] appears in the bibliography of [4], which carries [9] as the state of the art. Consequently $M(n)$ can be improved immediately and with no new mathematics: at $n = 4$ its inner argument drops from $R(K_4^*, L_3) \le 16$ to $r(I_4, L_3) = 15$, and m^2 may be replaced by $m^2 - m + 3$ throughout. With the value proved in Section 6 the first case of that constant becomes explicit rather than bounded. The source defines $M(n)$ for every integer $n \ge 2$, so $n = 2$ is the first case; since $r(I_2, L_3) = 4$ and $A_3(4) = 9$, one gets $M(2) = R(9,3) = 36$ — the upper half of that value being Grinstead and Roberts' [7], with both halves recorded in [12, Table Ia] — where the published data allowed only $M(2) \le R(R(4,4,4),3)$.

1.3. **What this paper proves.** Sections 3 and 5 reproduce the published material in the form the later sections consume: the m^2 bound, the sharp bound $m^2 - m + 3$, the values at $m \le 5$, and the analogous recursion for the L_4 column. The results we claim as new are these; where only one end of a bracket is new, the item says which.

- $A_3(4) = 9$ (Theorem 6.11), the first value of the avoidance number beyond the elementary $A_3(3) = 3$. The published bracket was $[1.125,\ R(4,4,4)]$. The lower half is an explicit 3-colouring of the arcs of K_8^* in which all 70 four-sets fail the condition; the upper half is an exhaustive search of nine-vertex colourings, made finite enough to run by the symmetry of Section 6.3.
- $A_3(4) \le 17$ (Theorem 6.5), the best bound we can prove with no search at all, from a new obstruction on *pairs* of vertices: in any $A_3(4)$ witness two distinct vertices send the same colour into at most four common heads, and that is sharp.

- $A_3(5) \geq 18$ (Theorem 6.12), from a 17-vertex colouring; the published lower bound is $3^3/2^4 = 1.6875$ and the published upper bound is the unknown $R(5,5,5)$. We also record that A_3 is non-decreasing, which the definition does not make obvious.
- $29 \leq r(I_6, L_3) \leq 33$ (Theorem 4.1). The upper end is [8]'s published bound at $m = 6$; what is new is the lower end, a 28-vertex circulant. [8] give no construction at $m = 6$, and the only lower bound that follows from their text is the general $r(I_6, L_3) \geq r(I_6, K_3) = 18$; we found none better elsewhere.
- $21 \leq r(I_3, L_4) \leq 25$ (Theorem 5.4). Again the upper end is [8]'s, being their own formula at $(m, n) = (3, 4)$; what is new is the lower end, a 20-vertex circulant. The Coda of [8] names $r(I_3, L_4)$ as the natural continuation of their work. Their text states no lower bound for this cell; what was available was the elementary $r(I_3, L_4) \geq r(I_3, L_3) = 9$.

We add three structural results that are not new statements about the numbers but are what makes the computations above feasible: the extremal structure of an $r(I_m, L_3)$ witness at $m^2 - 1$ vertices together with a relabelling that may be assumed without loss of generality (Proposition 3.6); a reduction of $A_3(4) = 9$ to two propositional refutations (Proposition 6.6); and the symmetry that makes those refutations cheap — the colours of A_3 may be permuted *separately at each vertex* (Proposition 6.7). Finally $A_3(3) = 3$ (Proposition 6.1) is elementary and is recorded because it is the first value of A_3 and fixes the convention $A_3(k) > k - 1$. For each of the results above we ran a literature search; each returned nothing, and a search returning nothing is evidence about the search rather than about the literature. The limits of ours are stated in Section 9.

Relation to the earlier version. An earlier version of this paper proved the bracket $9 \leq A_3(4) \leq 28$ and left the value open, reducing it to two propositional refutations we had not obtained. The present version determines $A_3(4) = 9$; the missing ingredient was the symmetry of Section 6.3, which the earlier version did not use. The two brackets for r are unchanged. Beyond the value, what is new here is Section 6.2 (the pair obstruction and $A_3(4) \leq 17$, both proved with no search) and Section 6.4 (A_3 is non-decreasing, and $A_3(5) \geq 18$). Sections 2–5 are unchanged as mathematics, but several attributions in them have been corrected against the sources; and one number is corrected: the earlier version quoted $[51, 62]$ as the published bracket for $R(4,4,4)$, which is in fact the published bracket for $R(3,3,3,3)$ [12]. Nothing in either version depends on that value.

2. Definitions

Throughout, a colouring of K_N^* is a function on ordered pairs of distinct vertices; the vertex set is $\{0, 1, \ldots, N-1\}$.

Definition 2.1 (the avoidance number)**.** Let c be a 3-colouring of the arcs of K_N^*. A set S of vertices is *good* for c if

$$\forall x \in S \;\; \exists j \;\; \forall y \in S \setminus \{x\} : \; c(y \to x) \neq j,$$

that is, if for every $x \in S$ the arcs *into* x from the other vertices of S miss at least one of the three colours. A colouring in which no k-set is good is a *k-witness*, and $A_3(k)$ is the least N such that no k-witness on N vertices exists.

The direction in Definition 2.1 is the one [4] states ("the edges from vertices in S to x") and it is not interchangeable with its mirror; see Proposition 7.1(iii).

Definition 2.2 (the directed Ramsey number)**.** Let c be a red/blue colouring of the arcs of K_N^*. A set S spans a *red* K_m^* if $|S| = m$ and $c(a \to b)$ is red for all distinct $a, b \in S$. Distinct vertices a, b, d form a *blue* L_3 if $a \to b$, $b \to d$ and $a \to d$ are all blue. A colouring containing neither is an *m-witness*, and $R(K_m^*, L_3)$ is the least N admitting no m-witness on N vertices.

A colouring is *oriented* on a vertex set V if no pair of distinct vertices of V carries two blue arcs. Reading a blue arc as an edge of an oriented graph and a doubly red pair as a non-edge, an m-witness that is oriented is exactly an $\{I_m, L_3\}$-free oriented graph in the sense of [8].

Lemma 2.3. $R(K_m^*, L_3) = r(I_m, L_3)$.

Proof. Oriented graphs are digraphs, so one inequality is trivial. For the other, given an m-witness c on V let c' agree with c except that on every pair $\{a, b\}$ with both arcs blue it reddens the arc from the larger vertex to the smaller. Then c' is oriented; it never turns a red arc blue, so it creates no blue L_3; and it leaves one arc of every doubly blue pair blue, so it creates no red K_m^* either. Hence c' is an oriented m-witness on the same vertex set. $\square$

From now on we write $r(I_m, L_3)$ and, for the transitive tournament on four vertices, $r(I_m, L_4)$: a blue L_4 is a quadruple a, b, d, e of distinct vertices with all six forward arcs ab, ad, ae, bd, be, de blue, and $r(I_m, L_4)$ is the least N such that every oriented graph on N vertices contains an independent m-set or a blue L_4. For a vertex v we write $N^+(v)$ and $N^-(v)$ for the blue out- and in-neighbourhoods and $I(v)$ for the set of vertices joined to v by two red arcs, so that

$$V = \{v\} \;\uplus\; N^+(v) \;\uplus\; N^-(v) \;\uplus\; I(v) \tag{1}$$

whenever the colouring is oriented on V.

Both quantities are described above by a threshold condition — a witness exists one step below and none at the value — and such a description determines at most one number.

Proposition 2.4. *Fix k (respectively m). If N and N' both have the property that some k-witness (respectively m-witness) on $N-1$ vertices exists while none exists on N vertices, then $N = N'$. Consequently Definitions 2.1 and 2.2 determine $A_3(k)$ and $r(I_m, L_3)$ uniquely.*

Proof. Restricting a witness on N vertices to the first $N' \le N$ of them is again a witness, so the set of N carrying a witness is downward closed; a threshold of a downward closed set of natural numbers is unique. $\square$

3. The L_3 column: the bounds and the values at $m \le 5$

This section is background. Nothing in it is claimed as new; the two bounds are re-derived from Definition 2.2 in the form the later sections use, and the values are those of [6, 1, 8].

3.1. **The quadratic bound.**

Proposition 3.1. *$r(I_m, L_3) \le m^2$ for every m. More precisely, if c is an m-witness on a vertex set V then $|V| + 1 \le m^2$.*

Proof. Fix $v \in V$ and let $A = N^+(v)$. For distinct $b, d \in A$, if $b \to d$ were blue then $v \to b$, $b \to d$, $v \to d$ would be a blue L_3; so $b \to d$ is red, and by the same argument with b and d exchanged so is $d \to b$. Hence A spans a red clique and $|A| \le m - 1$. Dually, for distinct $b, d \in B = N^-(v)$ a blue $b \to d$ would give the blue L_3 $b \to d \to v$; so $|B| \le m - 1$. Finally a red K^*_{m-1} inside $I(v)$ together with v would be a red K^*_m, so $I(v)$ carries an $(m-1)$-witness. Writing $g(m)$ for the largest witness size, $g(m) \le 1 + 2(m-1) + g(m-1)$ with $g(1) = 0$, whence $g(m) \le \sum_{i=2}^{m}(2i-1) = m^2 - 1$. □

This is Larson and Mitchell's bound [9]; the proof above was found from Definition 2.2 and is a re-derivation, not a transcription of theirs. Its value here is that the *upper* half of every exact value below $m = 4$ becomes free: an exact value needs only a witness on $m^2 - 1$ vertices.

3.2. **The sharp bound.**

Theorem 3.2 (Ihringer–Rajendraprasad–Weinert)**.** *For every $m \ge 3$, $r(I_m, L_3) \le m^2 - m + 3$. Moreover every $\{I_m, L_3\}$-free oriented graph on exactly $m^2 - m + 2$ vertices has at least $\frac{1}{2}(m^2 - m + 2)(2m - 3)$ edges.*

Proof sketch. The two statements are proved together by the induction of [8]: the bound at m gives the edge count at $m + 1$, and the two at m give the bound at $m + 1$.

For the edge count, if $|V| \ge m^2 + m + 2$ then $I(v)$ is $\{I_m, L_3\}$-free for every v, so by the bound at m it has at most $m^2 - m + 2$ vertices; by (1), $\deg(v) \ge |V| - 1 - (m^2 - m + 2) = 2(m + 1) - 3$. Summing over v and halving gives the claim.

For the bound, suppose $|V| = m^2 + m + 3$. Then (1) and the caps $|N^{\pm}(v)| \le m$, $|I(v)| \le m^2 - m + 2$ leave no slack, so every vertex has $|N^+(v)| = |N^-(v)| = m$ and $|I(v)| = m^2 - m + 2$. Fix v, put $J = I(v)$ and $N = N^+(v) \cup N^-(v)$. Counting the edges between N and J from below by the private-neighbour lemma (Lemma 3.3) and from above by the degree sum $|E(N, J)| + 2|E(J, J)| = 2m|J|$ yields $4|J| + 2|E(J, J)| \le 2m|J| + 4m$. The three cases of [8] differ only in the lower bound available for $|E(J, J)|$: at $m = 3$, $|J| = 8$ and the eight-vertex $\{I_3, L_3\}$-free graph has in- and out-degree 2 at every vertex, so $|E(J, J)| = 16$ and the inequality reads $64 \le 60$; at $m = 4$, $|J| = 14$ and a fourteen-vertex $\{I_4, L_3\}$-free oriented graph has at least 38 edges, giving $132 \le 128$; and for $m \ge 5$ the induction hypothesis on edges gives $|E(J, J)| \ge \frac{1}{2}(2m - 3)|J|$, forcing $m^2 - 5m + 2 \le 0$, false from $m = 5$ on. The general case therefore begins where the two special lemmas stop being needed, and the $m = 4$ lemma is unavoidable: the induction hypothesis alone gives only 35 edges on 14 vertices where 37 are required. □

The next lemma is the counting step of [8], stated here for an arbitrary independent set rather than for the two blue neighbourhoods of a vertex; the proof is theirs.

Lemma 3.3 (private neighbours)**.** *Let N be an independent set of size m in an $\{I_{m+1}, L_3\}$-free oriented graph and let J be disjoint from N. Then* $2|J| \leq |E(N, J)| + 2m$.

Proof. Every $x \in J$ has a neighbour in N, or $N \cup \{x\}$ would be independent of size $m+1$. Call x *private* if it has exactly one. A vertex $u \in N$ has at most two private neighbours: three of them x, y, z are either pairwise adjacent, and then $\{u, x, y, z\}$ is a tournament with no transitive triangle, which $r(I_2, L_3) = 4$ forbids; or contain a non-adjacent pair x, y, and then $\{x, y\} \cup (N \setminus \{u\})$ is independent of size $m+1$. So at most $2m$ vertices of J contribute one edge to $E(N, J)$ and the rest contribute at least two. □

3.3. **The values at $m \leq 5$.**

Theorem 3.4. $r(I_2, L_3) = 4$ *and* $r(I_3, L_3) = 9$.

Proof. The upper halves are Proposition 3.1 at $m = 2, 3$. For the lower halves: on three vertices, colour the arcs of the directed cycle $0 \to 2 \to 1 \to 0$ blue and the three reverse arcs red; a red K_2^* needs a pair with both arcs red and there is none, and a directed 3-cycle contains no transitive triple. On eight vertices there is a colouring with blue in- and out-degree exactly 2 at every vertex — the maximum the proof of Proposition 3.1 permits — and with no red K_3^*; the check is a finite verification over the $\binom{8}{3} = 56$ triples and the 8^3 ordered triples of vertices. Both values are also obtained here by a second, independent route: a propositional encoding of Definition 2.2 refuted by a SAT solver, with the refutation certificate checked by a verified checker. The two routes must name the same number by Proposition 2.4, and they do; this is the only available cross-check that the encoding expresses the definition. □

These are Erdős–Rado's and Bermond's values [6, 1]; [3] states the first in the ordinal form $R(\omega \cdot 2, 3) = \omega \cdot 4$.

Theorem 3.5. $r(I_4, L_3) = 15$. *In particular Larson and Mitchell's bound m^2 is not attained at $m = 4$.*

Proof sketch. The lower half is an explicit colouring of K_{14}^* with blue in- and out-degree exactly 3 at every vertex and no red K_4^*; the verification walks the $\binom{14}{4} = 1001$ four-sets and the 14^3 ordered triples. Proposition 3.1 gives only ≤ 16, so the value requires refuting the existence of a witness on 15 vertices. This was done by encoding Definition 2.2 at $N = 15$ as a propositional formula and adjoining the forced structure of Proposition 3.6, which fixes six arcs at vertex 0 and adds the derived degree clauses; the resulting formula has 36,883 clauses and is unsatisfiable, with a 21.2 MB refutation certificate checked inside the proof assistant by a verified certificate checker. Every clause of the narrowing is derived from Proposition 3.6 inside the same formal development, so the refutation of the narrowed formula is a refutation of the witness property with no side condition. □

The narrowing is what makes the cell reachable. Of five sound encodings of the same instance, differing only in which derived facts they assert, the definition alone was unfinished after an hour with a certificate past 2 GB, while the relabelling with the derived degree bounds took 4 s and 21.2 MB — a factor of at least 900 in time and 90 in certificate size, and the difference between a certificate a verified checker can hold and one it cannot.

Proposition 3.6. *Let c be an m-witness on V with $|V| = m^2 - 1$. Then for every $v \in V$: $|N^+(v)| = |N^-(v)| = m - 1$ exactly, $N^+(v)$ and $N^-(v)$ are disjoint — no pair of vertices has both its arcs blue — and $|I(v)| = (m-1)^2 - 1$, with $I(v)$ carrying an extremal $(m-1)$-witness. Moreover, if σ is injective on W and maps W into V then $c \circ \sigma$ is a witness on W; consequently, at $m = 4$ and $|V| = 15$ one may assume $N^+(0) = \{1, 2, 3\}$, $N^-(0) = \{4, 5, 6\}$ and $I(0) = \{7, \ldots, 14\}$.*

Proof. At $|V| = m^2 - 1$ the chain $|V| \le 1 + |N^+(v)| + |N^-(v)| + |I(v)| \le 1 + (m - 1) + (m-1) + ((m-1)^2 - 1)$ from the proof of Proposition 3.1 has no slack, so every inequality in it is an equality; disjointness is the statement that the union in (1) is disjoint, which the same count forces. Invariance under relabelling is immediate from Definition 2.2, both conditions being conditions on the image. For the last clause, the required σ is built explicitly from the sorted lists of the three blocks, so it is a function of the colouring with nothing chosen. $\square$

Proposition 3.7. *$r(I_4, L_3) = 15$ follows from Theorem 3.2 and the 14-vertex witness alone, without any refutation.*

Proof. $4^2 - 4 + 3 = 15$, so Theorem 3.2 gives ≤ 15 outright, and the 14-vertex witness gives ≥ 15. $\square$

Proposition 3.7 supersedes the certificate route as a proof; we keep both, since Proposition 2.4 forces the two to name the same number and an error of one anywhere in the induction of Theorem 3.2 would show up as a disagreement.

Theorem 3.8. *$r(I_5, L_3) = 23$.*

Proof. The upper half is Theorem 3.2 at $m = 5$. The lower half is the Cayley digraph on $\mathbb{Z}_{22}$ with connection set $D = \{1, 4, 10, 17\}$: the arc $a \to b$ is blue exactly when $b - a \in D$. Since D is sum-free in $\mathbb{Z}_{22}$ there is no blue transitive triangle, and $D \cap (-D) = \emptyset$ makes the graph oriented; the absence of a red K_5^* was verified by exhausting the $\binom{22}{5} = 26{,}334$ five-sets. $\square$

This is [8]'s value and their construction: they write the connection set out in the proof, as $\{x \mapsto x+1,\ x \mapsto x+4,\ x \mapsto x-5,\ x \mapsto x+10\}$ on $\mathbb{Z}_{22}$, and draw it in a figure. The same graph was found independently here by the sweep of Section 4 restricted to $m = 5$, before either was read; the agreement on all four elements of D — note $-5 \equiv 17$ — checks the transcription.

4. The bracket at $m = 6$

Theorem 4.1. *$29 \le r(I_6, L_3) \le 33$.*

Proof. The upper end is Theorem 3.2 at $m = 6$, where $6^2 - 6 + 3 = 33$; Larson and Mitchell's bound would give only 36. For the lower end, let c be the Cayley digraph on $\mathbb{Z}_{28}$ with connection set $D = \{1, 4, 12, 15, 22\}$. Then D is sum-free, so there is no blue L_3, and $D \cap (-D) = \emptyset$, so c is oriented. That c has no red K_6^* — equivalently, that the independence number of the underlying graph is at most 5 — was established by exhausting all $\binom{28}{6} = 376{,}740$ six-sets. $\square$

The lower end moves the elementary $r(I_6, L_3) \ge r(I_6, K_3) = 18$, which is all that was available: any oriented $\{I_6, K_3\}$-free graph is $\{I_6, L_3\}$-free.

For an exact value Theorem 3.2 asks for a witness on 32 vertices, and [8] give none past $m = 5$. The cheapest family to try is the one that succeeded at $m = 5$. A sweep over *every* sum-free connection set $D \subseteq \mathbb{Z}_n \setminus \{0\}$ of size 5 with $D \cap (-D) = \emptyset$, for $n = 26, \ldots, 33$, gives the following counts, the second column being the number of such D and the third the number of them whose Cayley digraph has independence number at most 5:

n	sum-free candidates	witnesses
26	1,104	12
27	2,232	18
28	2,886	6
29	4,256	0
30	5,288	0
31	8,250	0
32	10,368	0
33	15,794	0

The sweep was run twice with independent implementations, using different data structures and different independent-set searches, and the two agree on every row; run on the settled values it reproduces them, returning $n = 8$ with $D = \{1, 6\}$ at $m = 3$, $n = 13$ with $D = \{1, 3, 9\}$ and nothing at $n = 14$ for $m = 4$, and $n = 22$ with $D = \{1, 4, 10, 17\}$ and nothing at $n = 23$ for $m = 5$. The $m = 4$ row is the useful negative control: [8] independently remark that there is no oriented $\{I_4, L_3\}$-free Cayley graph on 14 vertices, so a published sentence confirms the sweep in its negative direction while the $m = 5$ row confirms it in its positive one.

So circulants reach 28 and stop, four short of what an exact value at $m = 6$ needs; whatever closes $m = 6$ will not be a circulant. The best circulant is one short of the truth at $m = 4$, exactly extremal at $m = 5$ — the case [8] published a figure for — and four short at $m = 6$.

5. The L_4 column

The Coda of [8] contains, verbatim, the sentence: "Determining $r(I_3, L_4)$ would continue our work and seems feasible given the size of the candidates for examples of (I_3, L_4)-free graphs." This section is that column.

5.1. **The recursion.**

Proposition 5.1. *Let c be an oriented colouring on V with no red K^*_m and no blue L_4, and let $v \in V$. Then $N^+(v)$ and $N^-(v)$ carry no red K^*_m and no blue L_3, and $I(v)$ carries no red K^*_{m-1} and no blue L_4. Consequently, writing*

$$c_3(m) = \begin{cases} 0, & m \le 1 \\ 3, & m = 2 \\ m^2 - m + 2, & m \ge 3 \end{cases} \qquad c_4(1) = 0, \quad c_4(m) = 1 + 2c_3(m) + c_4(m-1),$$

every such V satisfies $|V| \le c_4(m)$, and hence $r(I_m, L_4) \le c_4(m) + 1$ for every m. The bound holds for digraph colourings as well, with no orientedness hypothesis.

Proof. A blue L_3 inside $N^+(v)$ becomes a blue L_4 once v is prefixed, and a blue L_3 inside $N^-(v)$ becomes one once v is appended; a red K^*_m inside a neighbourhood is already a red K^*_m. A red K^*_{m-1} inside $I(v)$ together with v is a red K^*_m. Now split V by (1) and cap the three parts by $c_3(m)$, $c_3(m)$ and $c_4(m-1)$; the value $c_3(2) = 3$ is Proposition 3.1 at $m = 2$ and $c_3(m) = m^2 - m + 2$ for $m \ge 3$ is Theorem 3.2.

For the last clause, apply the orienting map of Lemma 2.3, which cannot create a blue L_4 either, since it only reddens arcs. □

The neighbourhood lemma and the recursion are [8]'s. We quote no constant into them: both inputs are the bounds of Section 3, so every vertex saved there is spent twice here, and Proposition 5.2 then checks the outcome against [8]'s own closed form for this column.

Proposition 5.2. *$c_4(2) = 7$, $c_4(3) = 24$, $c_4(4) = 53$, $c_4(5) = 98$ and $c_4(6) = 163$; hence*

$$r(I_2, L_4) \leq 8, \quad r(I_3, L_4) \leq 25, \quad r(I_4, L_4) \leq 54, \quad r(I_5, L_4) \leq 99, \quad r(I_6, L_4) \leq 164.$$

These agree with the closed formula $v(m,n) = \sum_{i=0}^{n-2} \binom{i+m-1}{i+1} 2^i - \binom{m+n-6}{m-4} 2^{n-3} + 1$ of [8], which bounds $r(I_m, L_n)$ for $m \geq 2$ and $n \geq 3$, at $n = 4$ and $m \leq 6$; the same formula reproduces $m^2 - m + 3$ at $n = 3$ and 2^{n-1} at $m = 2$.

Proof. Evaluation of the recursion, and of v, over the stated ranges. □

The agreement is a check on the transcription, not evidence that the formula is tight: at $m = 2$, $n = 5$ it gives 16 while the true value of $r(I_2, L_5)$ is 14.

5.2. **The values.**

Theorem 5.3. *$r(I_2, L_4) = 8$.*

Proof. The upper half is $c_4(2) = 7$, that is, $1+2(r(I_2, L_3)-1)+0$ with $r(I_2, L_3) = 4$. The lower half is the Paley tournament on $\mathbb{Z}_7$, the Cayley digraph with connection set the quadratic residues $\{1, 2, 4\}$: since -1 is a non-residue modulo 7 it is a tournament, so there is no red K_2^*, and a finite check over its vertex quadruples shows it has no blue L_4. □

This is Erdős and Moser's value [5] as cited in [8], reproduced not claimed.

Theorem 5.4. *$21 \leq r(I_3, L_4) \leq 25$.*

Proof. The upper end is Proposition 5.2. For the lower end, let c be the Cayley digraph on $\mathbb{Z}_{20}$ with connection set $D = \{2, 4, 5, 7, 11, 14\}$. Then $D \cap (-D) = \emptyset$, so c is oriented, and a finite check establishes that c has no independent 3-set and no blue L_4. The check is organised so that it stays inside the proof assistant's kernel: since every blue L_4 extends a blue L_3, it suffices to enumerate the 20^3 ordered triples and, for each of the few that form a blue L_3, the twenty candidate extensions, rather than all 20^4 quadruples. That this reformulation is equivalent is itself proved rather than assumed. □

What was available for this cell is $[9, 25]$: the upper end is Proposition 5.2, and the lower end is only the elementary $r(I_3, L_4) \geq r(I_3, L_3) = 9$, no lower bound for $r(I_3, L_4)$ being stated in [8]. The obvious construction, the lexicographic product of the Paley tournament on $\mathbb{Z}_7$ with I_2, reaches only 14 vertices. A sweep over every connection set $D \subseteq \mathbb{Z}_n \setminus \{0\}$ with $D \cap (-D) = \emptyset$ — that is, over every Cayley digraph on a cyclic group that is oriented — for $n = 3, \ldots, 24$ counts the $\{I_3, L_4\}$-free ones as follows:

n	14	15	16	17	18	19	20	21	22	23	24
witnesses	52	8	40	32	6	18	8	0	0	0	0

The range is the whole range that matters, since Theorem 5.4 caps a witness at 24 vertices. So 20 is the exact circulant maximum and exactly eight connection sets attain it. Moreover a 21-vertex witness contains a 20-vertex one, and for a *given* 20-vertex graph the one-vertex extension question is decidable by a search over the 3^{20} assignments of each old vertex to one of three states (new vertex dominated, dominating, or non-adjacent), pruned by two necessary conditions: no two non-adjacent old vertices may both be non-adjacent to the new one, and every blue L_3 of the old graph must have all four insertion slots blocked. Run on all eight extremal circulants the search answers no in every case, and it was validated by four positive controls. This does not prove $r(I_3, L_4) = 21$: a 21-vertex witness could sit above a non-circulant 20-vertex graph, and those were not enumerated.

6. The avoidance number

6.1. **The first value, and a lower bound for the second.**

Proposition 6.1. $A_3(3) = 3$.

Proof. If $|S| \leq 3$ then each $x \in S$ receives at most two arcs from $S \setminus \{x\}$, and two arcs cannot carry three colours, so some colour is missing and S is good. Hence no 3-witness exists once three vertices are available. Below three vertices there is no 3-set at all, so the witness condition holds vacuously and $A_3(3) > 2$. □

Against the published bracket $[0.75, 17]$ this is the whole value, but it is arithmetic rather than a result. It does fix a convention: $A_3(k) > k - 1$ always, because the witness condition is vacuous below k vertices.

Theorem 6.2. $A_3(4) \geq 9$.

Proof. The following 3-colouring of the arcs of K_8^*, given as a matrix whose (a, b) entry is the colour of $a \to b$ (diagonal entries are not read), has the property that *all* $\binom{8}{4} = 70$ four-sets are bad: for every four-set S some $x \in S$ receives all three colours from the other three vertices.

$$\begin{pmatrix} \cdot & 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & \cdot & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & \cdot & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & \cdot & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 & \cdot & 2 & 1 & 1 \\ 0 & 1 & 2 & 2 & 1 & \cdot & 1 & 0 \\ 2 & 2 & 2 & 1 & 2 & 0 & \cdot & 2 \\ 2 & 1 & 0 & 2 & 1 & 1 & 0 & \cdot \end{pmatrix}$$

The verification is the exhaustive check of those 70 four-sets against Definition 2.1. □

The colouring was found by a SAT solver, re-verified by an independent checker sharing no code with the encoder, and then checked a third time against Definition 2.1. The triple check is not ceremony: the first propositional encoding tried here carried at-least-one-colour clauses but no at-most-one clauses — the standard soundness-preserving simplification when a formula is used only for *refutation*, and one that is unsound in the satisfying direction, since an arc taking two colours at once makes a rainbow triple easier to produce — and it reported spurious witnesses at $N = 9$ and $N = 10$. The published lower bound at $k = 4$ is $3^2/2^3 = 1.125$.

6.2. **Two upper bounds proved with no search.** The first bound counts *rainbow triples*, one apex at a time. For a vertex x let $P_j(x)$ denote the set of vertices sending colour j into x, so that the three classes partition the other $N-1$ vertices, and write $n_j(x) = |P_j(x)|$.

Theorem 6.3. $A_3(4) \le 28$.

Proof. Let c be a 4-witness on N vertices. Every four-set S fails to be good, so some $x \in S$ receives all three colours from $S \setminus \{x\}$; since $|S \setminus \{x\}| = 3$, those three arcs are a *rainbow*, one vertex from each class of x. Thus S is recovered from the quadruple (x, y_0, y_1, y_2) with $y_j \in P_j(x)$, the four vertices being distinct because the y_j carry different colours into x. The resulting map is injective, so

$$\binom{N}{4} \;\le\; \sum_x n_0(x)\, n_1(x)\, n_2(x). \tag{2}$$

By the arithmetic–geometric mean inequality for three terms summing to $N-1$, each summand is at most $\big((N-1)/3\big)^3$, so $27\binom{N}{4} \le N(N-1)^3$. At $N = 28$ this reads $552{,}825 \le 551{,}124$, which is false. $\square$

At $N = 9$ the same inequality reads $126 \le 162$: it is nowhere near settling the exact value, and we record that explicitly because a bound that motivates a computation is exactly the kind of thing one is tempted to over-read. The slack has a structural source. Inequality (2) sees each four-set through a single vertex, and therefore cannot see any constraint that lives on *pairs* of vertices. There is one, and it is what the rest of this subsection is about. For distinct vertices u, v put

$$D(u,v) \;=\; \{\, x \notin \{u,v\} \;:\; c(u \to x) = c(v \to x) \,\},$$

the set of heads at which u and v land in the same in-class.

Theorem 6.4. *Let c be a 4-witness on N vertices. Then $|D(u,v)| \le 4$ for every pair of distinct vertices u, v, at every N. The bound is attained: the eight-vertex witness of Theorem 6.2 has $D(1,2) = \{3,4,5,7\}$.*

Proof. A vertex $x \in D(u,v)$ is never the apex of a four-set containing both u and v: the arcs into x from $\{u,v,y\}$ include the two equal colours $c(u \to x) = c(v \to x)$, so they carry at most two colours and x misses one. Hence for distinct $x, y \in D(u,v)$ the four-set $\{u,v,x,y\}$, which is bad, must be bad through u or through v: either u receives three different colours from $\{v,x,y\}$, or v receives three different colours from $\{u,x,y\}$. Write $\alpha(z) = c(z \to u)$ and $\beta(z) = c(z \to v)$, so the dichotomy says that for all distinct $x, y \in D(u,v)$ either

$$\begin{array}{llll} & \alpha(x) \ne \alpha(y), & \alpha(x) \ne \alpha(v), & \alpha(y) \ne \alpha(v) \\ \text{or} & \beta(x) \ne \beta(y), & \beta(x) \ne \beta(u), & \beta(y) \ne \beta(u). \end{array}$$

Nothing further about the colouring is used. If some $x_0 \in D(u,v)$ has $\alpha(x_0) = \alpha(v)$ then the first alternative fails against x_0, so the second holds for every $y \in D(u,v) \setminus \{x_0\}$; the two conditions $\beta(y) \ne \beta(u)$ and $\beta(y) \ne \beta(x_0)$ then pin β to a single colour on $D(u,v) \setminus \{x_0\}$, after which the first alternative forces α to be injective there, giving $|D(u,v)| \le 1 + 3 = 4$. Symmetrically if some x_0 has $\beta(x_0) = \beta(u)$. Otherwise α avoids $\alpha(v)$ and β avoids $\beta(u)$ on all of $D(u,v)$, while the dichotomy makes $z \mapsto (\alpha(z), \beta(z))$ injective; the image lies in a 2×2 grid, so $|D(u,v)| \le 4$ again. $\square$

The finite core of the argument was first checked by exhaustion: of the $3^2 \cdot 9^5 = 531{,}441$ local configurations of five candidate co-class vertices none survives the dichotomy, while 216 of the $3^2 \cdot 9^4$ four-element ones do. The bound is a genuine fact about witnesses rather than an artifact of the proof, by the tightness clause of Theorem 6.4.

Theorem 6.5. *$A_3(4) \leq 17$.*

Proof. Let c be a 4-witness on N vertices and count the triples (u, v, x) with $u, v \neq x$ and $c(u \to x) = c(v \to x)$ in two ways, the diagonal $u = v$ included. By the head, the count at a fixed x is $\sum_j n_j(x)^2$. By the pair, the count at a fixed (u, v) with $u \neq v$ is $|D(u, v)| \leq 4$ by Theorem 6.4, while at $u = v$ it is the number of $x \neq u$, namely $N - 1$; so the total is at most $N(N-1) + 4N(N-1) = 5N(N-1)$. By Cauchy–Schwarz, $3 \sum_j n_j(x)^2 \geq \left(\sum_j n_j(x)\right)^2 = (N-1)^2$ for every x, so the total is at least $N(N-1)^2/3$. Chaining, $N(N-1)^2 \leq 15N(N-1)$, that is $N \leq 16$. At $N = 17$ the two sides read $17 \cdot 256 = 4352$ against $3 \cdot 5 \cdot 17 \cdot 16 = 4080$. $\square$

The bound is the exact strength of this argument: at $N = 16$ the same two counts read $3600 \leq 3600$, so 16 survives it, and pushing lower needs the equality analysis rather than a better constant. Theorem 6.5 is superseded numerically by Theorem 6.11 below; we keep it because it is the strongest bound on $A_3(4)$ we can prove with no search of any kind, and it is the one that survives if a reader declines to accept a machine-checked exhaustive computation.

6.3. **The symmetry, and the value.** Determining $A_3(4)$ means refuting the existence of a 4-witness on nine vertices, and the space of nine-vertex colourings has 3^{72} elements. Two reductions cut it. The first is the counting bound.

Proposition 6.6. *There are two explicit propositional formulas, each of* 1,934 *clauses in* 6,542 *variables, such that if both are unsatisfiable then $A_3(4) \leq 9$ and hence, with Theorem 6.2, $A_3(4) = 9$.*

Proof sketch. Let c be a 4-witness on nine vertices, and write $p(x)$ for the product $n_0(x)n_1(x)n_2(x)$. Inequality (2) gives $\binom{9}{4} = 126 \leq \sum_x p(x)$ over nine summands, each at most 18 — the maximum of abc subject to $a + b + c = 8$ — so some vertex x_0 has $p(x_0) \geq \lceil 126/9 \rceil = 14$. Over $a + b + c = 8$ the attainable products are $\{0, 6, 10, 12, 16, 18\}$, and only $\{3, 3, 2\}$ (product 18) and $\{4, 2, 2\}$ (product 16) reach 14. Both vertex names and colour names carry no information — Definition 2.1 is invariant under relabelling the vertices by any injection, and under permuting the three colours, since it asks each $x \in S$ to miss *some* colour and never names which — so one may assume $x_0 = 0$ and that its three in-classes are the initial intervals of sizes $(3, 3, 2)$ or $(4, 2, 2)$. That fixes eight arcs. The two formulas are the propositional encoding of Definition 2.1 at $N = 9$ together with the eight corresponding unit clauses, and the relabelling is constructed explicitly from the sorted in-class lists, so nothing is chosen and the reduction is a proof rather than a convention. $\square$

Proposition 6.6 on its own does not settle the value, and the obstruction is certificate size rather than solver time. Refutation certificates for these two formulas grow at roughly 120 MB per minute while the verified checker available to us tops out near 1.5 GB, so the cell carries a budget of about thirteen minutes of solving; both formulas were still unresolved long past it, as was a further sound narrowing that sorts within the three blocks, and the unnarrowed nine-vertex formula had earlier been abandoned after 2 h 40 m with a certificate of 4.83 GB.

What was missing is a symmetry of A_3 larger than the one used above. Permuting the three colours globally is a group of order 6; the true group is much larger, because Definition 2.1 never compares the colour of an arc into x with the colour of an arc into y.

Proposition 6.7. *Let c be a k-witness on N vertices and let $p_x : \{0,1,2\} \to \{0,1,2\}$ be surjective for each vertex x. Then $c'(a \to b) = p_b\big(c(a \to b)\big)$ is again a k-witness.*

Proof. Let S be good for c' and let $x \in S$ miss the colour j among the arcs of c' into x. Choose i with $p_x(i) = j$, which exists because p_x is surjective. If some $y \in S \setminus \{x\}$ had $c(y \to x) = i$ then $c'(y \to x) = p_x(i) = j$, a contradiction; so x misses i under c. Hence S is good for c, and a colouring with no good k-set is carried to a colouring with no good k-set. □

The group is thus $(S_3)^N$, of order $6^9 = 10{,}077{,}696$ at $N = 9$, and it can be quotiented out completely, one column of the arc matrix at a time.

Definition 6.8. A colouring c of K_N^* is *column-canonical* if for every vertex x, reading the colours $c(u \to x)$ with $u \neq x$ in increasing order of u gives a restricted growth string: the first entry is colour 0, and colour 2 does not occur before colour 1 has occurred.

Proposition 6.9. *There is an explicit map sending each colouring c to a column-canonical colouring of the shape of Proposition 6.7, hence carrying witnesses to witnesses; and on a column that is already a restricted growth string, the permutation it applies is the identity.*

Proof. At each column x let a be the colour the least vertex $u \neq x$ sends into x, and let b be the colour sent by the least vertex whose colour differs from a, or $a+1$ modulo 3 if the column is constant; then $b \neq a$ unconditionally, and p_x is the permutation sending $a \mapsto 0$, $b \mapsto 1$ and the third colour to 2. The resulting column is a restricted growth string, and Proposition 6.7 applies. If the column already is one, then $a = 0$; and $b = 1$, since the least vertex sending a colour other than 0 cannot send colour 2 without being preceded by a vertex sending colour 1, which would contradict its minimality. So p_x fixes 0 and 1, hence is the identity. □

Remark 6.10. The canonicalization is complete rather than partial: the colours into x partition the other $N-1$ vertices into at most three blocks, and each such partition has exactly one restricted growth labelling. The last clause of Proposition 6.9 is what allows the two reductions to be composed *in this order* — relabel the vertices first, recolour second. After the relabelling of Proposition 6.6, column 0 reads $0, \dots, 0, 1, \dots, 1, 2, \dots, 2$, which is already a restricted growth string, so the recolouring leaves it alone and the eight pinned arcs survive. The order matters: relabelling the vertices changes which entry of a column comes first, hence changes that column's normalization. Both admissible profiles have $n_0, n_1 \geq 1$, which is what makes the pinned column contain all three colours and so forces the identity above.

Definition 6.8 is not a new device. Asking that the first occurrences of the values $v_1, \dots, v_m$ appear in that order along a sequence is the *value precedence* constraint of Law and Lee [10], the standard way to break the symmetry of interchangeable values in constraint programming; Walsh [13] gives the encoding into a chain of ternary constraints and proves it enforces generalized arc consistency. At $m = 3$ values the constraint is exactly "restricted growth string". What is ours here is only that its soundness for A_3 is Propositions 6.7 and 6.9, proved inside the formal development and composed there with the vertex relabelling, rather than asserted outside it as a property of the search.

Theorem 6.11. $A_3(4) = 9$.

Proof sketch. The lower half is Theorem 6.2. For the upper half, let c be a 4-witness on nine vertices. By Proposition 6.6 we may assume that vertex 0 has in-class sizes $(3, 3, 2)$ or $(4, 2, 2)$ laid out on the initial intervals, and by Propositions 6.7 and 6.9 we may then assume in addition that c is column-canonical, the second assumption not disturbing the first. Column-canonicity is expressible by two unit clauses and $N - 2$ implications per column, so by 81 clauses at $N = 9$; adjoining them to the two formulas of Proposition 6.6 gives two formulas of 2,015 clauses in the same 6,542 variables. Both are unsatisfiable. Hence no 4-witness on nine vertices exists, and $A_3(4) = 9$ by Proposition 2.4. □

The two refutations are the only exhaustive computation the value rests on, and they are complete searches of the narrowed space, not samples: unsatisfiability of the two formulas says that no assignment whatever satisfies them, and the two propositions above are what turn that into a statement about all nine-vertex colourings. The narrowing is what makes them cheap. The same instances without the 81 clauses are the ones abandoned above; with them, a SAT solver refutes them in 2.3 s and 3.2 s, producing refutation certificates of 13.4 MB and 11.4 MB, both about one per cent of the certificate budget and both checked by a verified checker inside the proof assistant. Three further solver configurations were tried in search of a smaller certificate and all were worse.

Since a symmetry reduction that is unsound in the satisfying direction had already produced spurious witnesses here once (see the discussion after Theorem 6.2), the narrowing was validated where the answer is known before it was believed where it is not; the checks are collected in Proposition 7.2.

6.4. **A_3 is a ladder, and the first bound at $k = 5$.** Nothing in Definition 2.1 makes it obvious that A_3 is monotone in k; it is, and the reason is that goodness is inherited by subsets.

Theorem 6.12. (i) *A subset of a good set is good; consequently a k-witness is a $(k+1)$-witness, and $A_3(k) \leq A_3(k+1)$ for every k.*
(ii) *$A_3(5) \geq 18$.*
(iii) *$A_3(5) \geq 15$, by a colouring of K^*_{14} whose verification is a kernel computation carrying no compiled evaluation.*

Proof. (i) If $T \subseteq S$ and $x \in T$ misses the colour j among the arcs into x from $S \setminus \{x\}$, it misses j among the fewer arcs it receives from $T \setminus \{x\}$. So a good $(k+1)$-set would contain a good k-set, and a colouring with no good k-set has no good $(k+1)$-set. Already this gives $A_3(5) \geq 9$, from the eight-vertex colouring of Theorem 6.2 at no further cost.

(ii) and (iii) are the colourings below, for which the exhaustive checks over the $\binom{17}{5} = 6188$ and $\binom{14}{5} = 2002$ five-sets return no good set. □

The colouring for (ii), in the convention of Theorem 6.2:

$$\begin{pmatrix}
\cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & \cdot & 1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & \cdot & 2 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
1 & 2 & 0 & 2 & \cdot & 2 & 1 & 0 & 0 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 2 \\
2 & 0 & 0 & 0 & 0 & \cdot & 0 & 2 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 & \cdot & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & \cdot & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 1 \\
1 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & \cdot & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
2 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & \cdot & 0 & 0 & 2 & 2 & 2 & 1 & 2 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & \cdot & 2 & 0 & 0 & 2 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & \cdot & 0 & 2 & 2 & 2 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 2 & 1 & \cdot & 1 & 0 & 2 & 0 \\
1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & \cdot & 1 & 0 & 0 \\
0 & 2 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 2 & \cdot & 1 & 2 \\
2 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & \cdot & 2 \\
2 & 2 & 0 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 0 & 0 & 2 & \cdot
\end{pmatrix}$$

and the colouring for (iii):

$$\begin{pmatrix}
\cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & \cdot & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
2 & 1 & 0 & \cdot & 2 & 1 & 2 & 2 & 0 & 1 & 0 & 2 & 1 & 0 \\
0 & 2 & 2 & 1 & \cdot & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & 1 \\
0 & 2 & 2 & 2 & 0 & \cdot & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 2 & 0 & \cdot & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 2 & 2 & 2 & 0 & 2 & \cdot & 2 & 1 & 2 & 1 & 2 & 2 \\
0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & \cdot & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 2 & \cdot & 2 & 1 & 2 & 2 \\
1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 2 & 2 & \cdot & 2 & 1 & 2 \\
2 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & \cdot & 2 & 1 \\
2 & 1 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & \cdot & 2 \\
1 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & \cdot
\end{pmatrix}$$

Both were found by a SAT solver on the $k = 5$ analogue of the encoding of Proposition 6.6, narrowed by Propositions 6.7 and 6.9, which are symmetries of $A_3(k)$ at every k; the 17-vertex model was then decoded by a program sharing no code with the encoder, re-checked by a second, independently written checker, and checked a third time against Definition 2.1, which is the check the theorem depends on. Two witnesses are given because the two verifications have different standing: the $\binom{17}{5} = 6188$ subsets are past what the proof assistant's kernel will evaluate at a sane price — measured at 13.8 GB and unfinished after 12 minutes — so (ii) is verified by compiled evaluation, while (iii) is a kernel computation and carries no such assumption. A reader who declines compiled evaluation therefore still keeps $A_3(5) \geq 15$.

We do not know $A_3(5)$. Instances at $N \leq 16$ are satisfiable in under a second each and $N = 17$ took 11.1 s, but $N = 18$ returned no answer in 15 minutes and we stopped there, so 18 is a lower bound and not a value. The upper end is worse: the published bound is $A_3(5) \leq R(5, 5, 5)$, which is unknown, its best published lower bound alone being 454 [12], and neither of the two arguments of Section 6.2 reaches $k = 5$. Both are $k = 4$ phenomena, for the same structural reason: the failure of a four-set forces three vertices to be a *transversal* of the three in-classes of a fourth, a rare event, whereas at $k = 5$ inequality (2) asks four vertices merely to *meet* all three classes, which most four-sets do, so its constraint weakens with k while $\binom{N}{k}$ weakens faster — the two sides never cross for $N \leq 80$, the ratio growing rather than shrinking — and an apex now receives at least four arcs, among which one repeated colour is harmless. In summary,

$$A_3(3) = 3, \qquad A_3(4) = 9, \qquad 18 \leq A_3(5) \leq R(5, 5, 5),$$

and an upper bound at $k = 5$ needs a genuinely different argument.

7. Controls

Two things can go wrong above that no amount of internal consistency would reveal: the definitions could have been transcribed wrongly, and a threshold statement could be satisfied vacuously. A third is specific to Section 6.3: a narrowing could be over-constrained, in which case the refutations would be refutations of nothing. The following are checks against all three, each performed on explicit data.

Proposition 7.1. (i) No claimed value is too small. *For every $N \leq 14$, $r(I_4, L_3) \neq N$; for every $N \leq 8$, $A_3(4) \neq N$; $r(I_2, L_3) \neq 3$, $r(I_3, L_3) \neq 8$, $A_3(3) \neq 2$. Each is refuted by the corresponding witness alone, with no second search.*

(ii) No claimed value is too large. *For $m = 2, 3, 4$ and every $N > m^2$, $r(I_m, L_3) \neq N$, by Proposition 3.1; and $A_3(3) \neq N$ for every $N > 3$, by Proposition 6.1.*

(iii) The direction of the arcs in Definition 2.1 is load-bearing. *On the very grid of Theorem 6.2, the transposed reading — arcs* from *x to the vertices of S — makes 21 of the 70 four-sets good, where the reading of [4] makes none good.*

(iv) Transitivity in Definition 2.2 is load-bearing. *The blue directed 3-cycle used in Theorem 3.4 is not a blue L_3, while the all-blue triple on the same three vertices is; reading L_3 cyclically would refute that witness.*

(v) A red K_m^* needs both arcs. *Under the colouring "red exactly when $a < b$" every pair carries a red arc and yet there is no red K_2^*.*

(vi) Perturbation. *Recolouring the single arc $0 \to 1$ destroys both the 14-vertex witness of Theorem 3.5 and the 8-vertex one of Theorem 6.2.*

(vii) Tightness. $r(I_2, L_3) = 2^2$ *and* $r(I_3, L_3) = 3^2$*, so no bound $f(m) < m^2$ holds at every m, which is why the improvement to $m^2 - m + 3$ starts at $m = 3$.*

Proposition 7.2. (i) The narrowing of Section 6.3 is not over-constrained. *With the same 81-clause family and the profile clauses appropriate to $N = 8$, the formula is satisfiable; its model, decoded and read back as a colouring of K_8^*, is a genuine 4-witness — all 70 four-sets bad against Definition 2.1 — and it is column-canonical. An over-constrained narrowing could not produce one.*

(ii) The narrowing is not vacuous. *The eight-vertex witness of Theorem 6.2 is* not *column-canonical, so the 81 clauses do cut.*

(iii) Canonicalization on data. *Applying the map of Proposition 6.9 to that witness leaves a witness, and the result is column-canonical.*

(iv) The value is exactly one number. *$A_3(4) \neq 8$, $A_3(4) \neq 10$, and $A_3(4) \neq 27$, the last of these a value that Theorem 6.3 still permitted.*

The corresponding control for Theorem 6.12 is that no $N \leq 17$ is $A_3(5)$, which follows from the 17-vertex colouring and Proposition 2.4, and the corresponding control for Theorem 6.4 is its tightness clause.

Proposition 7.3. (i) L_4 is the transitive tournament and not the path. *The blue directed 5-cycle carries the blue path $0 \to 1 \to 2 \to 3$ but has $0 \to 2$*

red, and contains no blue L_4 at all; a path reading of L_4 would have refuted it.

(ii) The L_4 column is not the L_3 column in disguise. *The 20-vertex witness of Theorem 5.4 does contain a blue L_3, and L_3-freeness at $m = 3$ dies at nine vertices while L_4-freeness survives twenty.*

(iii) The bracket is exactly a bracket. *Any N with $r(I_3, L_4) = N$ satisfies $21 \le N \le 25$, and no value outside is permitted by what is proved.*

(iv) Both inputs to the recursion are load-bearing. *Driving Proposition 5.1 with $m^2 - 1$ instead of $m^2 - m + 2$ gives $c_4(4) = 55$ and $c_4(6) = 175$ in place of 53 and 163; using 4 rather than the true cap 3 at $m = 2$ gives $c_4(3) = 26$, which would miss $v(3, 4) = 25$.*

(v) Orientedness is not vacuous. *The all-blue colouring on two vertices avoids everything asked of it and is not oriented, which is why Proposition 5.1 orients rather than assumes.*

8. Verification

Every statement in this paper — the definitions, the two bounds of Section 3, the recursion of Section 5, each exact value and each bracket, the reduction of Proposition 6.6, the symmetry of Section 6.3, and all of the controls of Section 7 — has been formalised and machine-checked in Lean 4 against *mathlib* [14], with no unproved assumptions. Each finite verification is a computation inside the proof assistant's kernel, except where a kernel computation exhausts the available memory — the 22- and 28-vertex witnesses of Theorems 3.8 and 4.1, and the 17-vertex witness of Theorem 6.12(ii) — in which case it is a compiled evaluation; Theorem 6.12(iii) is the kernel-checked fallback for the last of these. The refutation certificates used in Theorems 3.4, 3.5 and 6.11 are checked by a verified certificate checker, and every step between a certificate and the statement it supports — the propositional encodings, their fidelity to the definitions, the relabelling of Proposition 3.6, and the two reductions of Section 6.3 — is proved in the kernel. In particular Propositions 6.7 and 6.9 are theorems of the development and not side conditions on the search. The encodings are printed by walking the formal definitions themselves. The development is a private repository; repository reference to be added at submission.

9. Remarks on the searches

For the results above we searched a local full-text mirror of the arXiv in the notations of both communities, the citation graphs of [8] and [4] available to us, the OEIS, and the standard dynamic survey of small Ramsey numbers [12] — which explicitly excludes directed and oriented Ramsey numbers from its scope, so its silence is a scope decision and not evidence. All versions of [8] were read in full; its Coda lists $r(I_3, L_4)$ as an open problem and does not mention $m = 6$.

For A_3 the search was run twice, the second time with the mirror at 881,145 papers covering announcements through August 2026, over 81 fixed-string patterns: the notation of [4], the phrase "avoidance number" and its variants, every phrasing we could think of for "the arcs into x miss a colour", and the published bound $3^{k-2}/2^{k-1}$ itself, which has exactly one occurrence in the corpus — [4], which served as a positive control for every query. We found no numeric value or bound of $A_3(k)$ beyond the two [4] prints, and no equivalent concept. The nearest published relatives were checked and are different objects: the colour-avoiding Ramsey number of Erdős–Hajnal–Rado and Erdős–Szemerédi is the uniform undirected version to which the source's own proof of $A_3(k) \leq R(k,k,k)$ reduces, while $A_3(k)$ is the weaker per-vertex relaxation in which each $x \in S$ may miss a *different* colour; local Ramsey numbers restrict the colouring and conclude monochromatic; the notation $A_k(n;r,s)$ of [11] is a different object — the least N such that every r-colouring of $[N]^{(k)}$ contains a tight monotone path on n edges using at most s colours, so the count is over a whole path rather than per vertex; and the remaining hits for "avoids at least one colour" cluster in reverse mathematics, where the objects are infinite and unordered.

What we could not check, and therefore do not claim as a clean negative: [9] and [1] themselves, known to us only through [8] and [3]; the pre-arXiv literature generally, including [6] and [5], which is the likeliest hiding place for A_3 if it has been studied; reviewing databases and citation indices to which we have no access; books, theses and non-arXiv venues; and anything appearing after August 2026. Finally, one methodological point: no fixed-string search in the notation of [4] could have located the settled table for $R(K_m^*, L_3)$, the published object carrying a different name and a different formulation, so for a quantity defined in terms of a Ramsey-type object the other community's notation is worth searching first.

References

[1] J.-C. Bermond, *Some Ramsey numbers for directed graphs*, Discrete Math. **9** (1974), no. 4, 313–321.

[2] L. Boza, *New upper bounds for the classical Ramsey numbers* $R(4,4,4)$*,* $R(3,4,5)$ *and* $R(3,3,6)$, arXiv:2603.10851v2 (2026).

[3] A. E. Caicedo and J. Hilton, *Topological Ramsey numbers and countable ordinals*, Foundations of Mathematics, Contemp. Math. **690**, Amer. Math. Soc., Providence, RI, 2017, pp. 87–120; arXiv:1510.00078.

[4] N. Duman, Ö. Gönül, B. Kaya, J. Saxena and Y. Tamer, *On closed Ramsey numbers of small countable ordinals*, arXiv:2604.23433v1 (25 April 2026).

[5] P. Erdős and L. Moser, *On the representation of directed graphs as unions of orderings*, Publ. Math. Inst. Hung. Acad. Sci., Ser. A **9** (1964), 125–132.

[6] P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.

[7] C. Grinstead and S. Roberts, *On the Ramsey numbers* $R(3,8)$ *and* $R(3,9)$, J. Combin. Theory Ser. B **33** (1982), 27–51.

[8] F. Ihringer, D. Rajendraprasad and T. Weinert, *New bounds on the Ramsey number* $r(I_m, L_n)$, arXiv:1707.09556; Discrete Math. **344** (2021), no. 3, Paper 112268.

[9] J. A. Larson and W. J. Mitchell, *On a problem of Erdős and Rado*, Ann. Comb. **1** (1997), no. 3, 245–252.

[10] Y. C. Law and J. H. M. Lee, *Global constraints for integer and set value precedence*, Principles and Practice of Constraint Programming — CP 2004, Lecture Notes in Comput. Sci. **3258**, Springer, Berlin, 2004, pp. 362–376.

[11] E. Mulrenin, C. Pohoata and D. Zakharov, *Color avoidance for monotone paths*, Discrete Anal. **2025**, Paper No. 23, 14 pp.; arXiv:2411.19823.

[12] S. P. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin., Dynamic Survey DS1, revision 18 (24 April 2026), doi:10.37236/21.

[13] T. Walsh, *Symmetry breaking using value precedence*, ECAI 2006, 17th European Conference on Artificial Intelligence, Frontiers in Artificial Intelligence and Applications **141**, IOS Press, 2006, pp. 168–172.

[14] The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, New York, 2020, pp. 367–381.