HERMITIAN SELF-DUAL MDS CODES FOR ABSOLUTELY MAXIMALLY ENTANGLED STATES: EXACT COUNTS AND MONOMIAL CLASSIFICATION

KOREA SUPERINTELLIGENCE LABS

Abstract. A Hermitian self-dual MDS code $[2k, k, k+1]_{q^2}$ produces, through the stabilizer construction, an absolutely maximally entangled state on $2k$ parties of local dimension q, and such codes are exactly the $k \times k$ matrices A over $\mathbb{F}_{q^2}$ with $A\overline{A}^{\mathsf{T}} = -I_k$ all of whose square submatrices are nonsingular. We determine the exact number of these codes in nineteen cells, including 30 569 011 200 codes $[8, 4, 5]_{25}$, 2 642 411 520 codes $[10, 5, 6]_9$, and none at all in the cell $[8, 4, 5]_9$; at $k = 2$ the count is $(q-2)(q+1)^3$ for every q we reach. We classify thirteen cells up to monomial equivalence: the $[8, 4, 5]_{25}$ cell has exactly seven classes, the $[10, 5, 6]_9$ cell exactly one, and the $[6, 3, 4]_{q^2}$ cells have $1, 1, 4, 8, 30, 51$ classes for $q = 2, 3, 5, 7, 11, 13$. Finally we answer, for the $[12, 6, 7]_{25}$ code constructed by Bevins and Bidav (arXiv:2608.05781), the equivalence question their paper leaves open: its monomial automorphism group has order 108, its class contains exactly 9 654 465 016 627 200 codes, and its Galois image is a second, inequivalent class, so the classification of that cell is nontrivial. Every numerical statement below is machine-checked in Lean 4.

1. Introduction

An *absolutely maximally entangled* state $\mathrm{AME}(n, q)$ is a pure state of n parties of local dimension q every one of whose reductions to at most $\lfloor n/2 \rfloor$ parties is maximally mixed [11, 12]; equivalently [11, Prop. 3], it is a pure quantum error-correcting code $[\![n, 0, \lfloor n/2 \rfloor + 1]\!]_q$. Deciding existence is in general a question about a unit vector in $(\mathbb{C}^q)^{\otimes n}$ and is not a finite problem. What is finite is the classical-coding sufficient condition that the literature actually runs on. An $\mathrm{AME}(n, q)$ state of *minimal support* — one carried, up to local unitaries, by $q^{\lfloor n/2 \rfloor}$ computational basis states, the fewest possible — exists if and only if an $(n, q^{\lfloor n/2 \rfloor}, \lfloor n/2 \rfloor + 1)_q$ MDS code exists; the direction from codes to states is Helwig and Cui's [13], and the equivalence is stated by Goyeneche, Alsina, Latorre, Riera and Życzkowski [14, Sec. 4]. An $\mathbb{F}_{q^2}$-linear Hermitian self-dual MDS $[2k, k, k+1]_{q^2}$ code gives a pure stabilizer code $[\![2k, 0, k+1]\!]_q$, and hence an $\mathrm{AME}(2k, q)$ state, by the Hermitian construction of Ketkar, Klappenecker, Kumar and Sarvepalli [15, Cor. 19] (for prime q already in [17]; see also [16]). The two routes are not the same: Proposition 22 below exhibits a q at which the Hermitian route reaches $\mathrm{AME}(2k, q)$ and the minimal-support route provably cannot. Finally, shortening the code carries an $\mathrm{AME}(2k, q)$ state of minimal support to an $\mathrm{AME}(2k-1, q)$ one [14, Sec. 4], and [1] reaches its odd-length states by projecting one party. None of these transfers is used as a premise below: every statement we prove is a statement about a matrix over a finite field.

Date: August 30, 2026.

Bevins and Bidav [1] recently used this route to settle five open cells, exhibiting Hermitian self-dual MDS codes $[12, 6, 7]_{25}$, $[18, 9, 10]_{121}$ and $[18, 9, 10]_{169}$ and hence AME$(12, 5)$, AME$(18, 11)$, AME$(18, 13)$ and their one-party projections. Their Criterion 1 — restated as Proposition 1 below — turns the search for such a code into a search for a $k \times k$ matrix, and it is the criterion the present paper works with throughout. Their paper closes with

> "The searches were not exhaustive, so classification and equivalence questions remain open."

This paper settles the counting and classification questions in every cell a complete enumeration can reach, and settles the equivalence question for the code Bevins and Bidav actually constructed.

Results. Write $\mathcal{H}(q, k)$ for the set of Hermitian self-dual MDS $[2k, k, k+1]_{q^2}$ codes and $h(q, k) = |\mathcal{H}(q, k)|$.

- **Exact counts** (Theorem 5, Corollary 6 and Proposition 23). We compute $h(q, k)$ exactly in the nineteen cells of Table 1. The two largest are $h(5, 4) = 30\,569\,011\,200$ (the AME$(8, 5)$ cell) and $h(3, 5) = 2\,642\,411\,520$ (the AME$(10, 3)$ cell); the informative zero is $h(3, 4) = 0$, no Hermitian self-dual MDS $[8, 4, 5]_9$ code at all, although $[8, 4, 5]_9$ MDS codes do exist and the *longer* cell $[10, 5, 6]_9$ is not empty. At $k = 2$ the answer is $(q-2)(q+1)^3$ at every q we reach.
- **Classification up to monomial equivalence** (Theorem 13). Thirteen cells are classified. The $[8, 4, 5]_{25}$ cell has exactly seven classes, with automorphism groups of orders $72, 48, 36, 18, 12, 12, 6$; the $[10, 5, 6]_9$ cell has exactly one; the $[6, 3, 4]_{q^2}$ cells have $1, 1, 4, 8, 30, 51$ classes for $q = 2, 3, 5, 7, 11, 13$.
- **The class of the Bevins–Bidav $[12, 6, 7]_{25}$ code** (Theorem 16). Its monomial automorphism group has order 108; its monomial equivalence class contains exactly $12! \cdot 6^{11}/18 = 9\,654\,465\,016\,627\,200$ codes; and its entrywise Galois image is a Hermitian self-dual MDS $[12, 6, 7]_{25}$ code lying in a *different* monomial class. So that cell has at least two classes — the two fuse under semilinear equivalence — and the classification question the source leaves open has a nontrivial answer.

What is new and what is not. Whether a cell is empty is, in most cases, already on record. A nonempty cell $\mathcal{H}(q, k)$ produces an AME$(2k, q)$ state, and each of the fifteen AME$(2k, q)$ states our nonempty cells produce is listed as existing in the table of AME states maintained by Huber and Wyderka [20] — consulted 30 August 2026, in its version of 8 June 2026 — except the two at $q = 13$, a local dimension that table does not reach; the quantum MDS parameters are also tabulated in [7]. Of our four empty cells, three sit under nonexistence entries of the same table — AME$(4, 2)$ and AME$(8, 3)$, both "No (Shadow)", and AME$(8, 2)$, "No (Scott)" — while the fourth, $h(4, 4) = 0$, is a statement about $\mathbb{F}_{16}$-linear codes that leaves the table's blank at the open cell AME$(8, 4)$ untouched (Remark 25). What is new is the exact cardinalities, which we did not find in the literature, in OEIS, or in an electronic-preprint corpus search. Three entries of our classification recover published facts and serve as controls rather than as results: the $[6, 3, 4]_4$ cell is the classical hexacode cell, and our machinery returns its automorphism group of order $1080 = |3 \cdot A_6|$; the $[6, 3, 4]_9$ and $[10, 5, 6]_9$ cells are unique, matching Danielsen's

classifications of self-dual additive codes [5, 6]. The emptiness $h(3,4)=0$ likewise has published content — AME$(8,3)$ is excluded for arbitrary states by the shadow bound of Huber, Eltschka, Siewert and Gühne [10] — although we did not find the classical phrasing "there is no Hermitian self-dual MDS $[8,4,5]_9$ code" anywhere; our derivation is direct and independent. The classifications of $[8,4,5]_{25}$ and of $[6,3,4]_{q^2}$ for $q \geq 7$, the exact counts, and the automorphism order, class size and second class of Theorem 16 appear to be new.

These negative findings deserve their limits stated. We searched an electronic-preprint corpus, the arXiv API, OEIS and the web; none of the cardinalities or class numbers above, and none of the sequences they form, occurs in any of them, and the one published statement we found in the direction of our classification claims supports them: Danielsen writes that "to our knowledge, no complete classification of Hermitian self-dual *linear* codes over GF(9) have appeared so far" [6, Sec. I], and his own additive classification does not reach length 8 over GF(25) even as a class of larger codes [5, Table 2]. Two items we could not obtain and therefore cannot exclude: Bouyuklieva [8] announces a classification of Hermitian self-dual MDS $[8,4,5]_{64}$ codes — the same shape one field further on, at $q=8$, which is not a cell of Table 1 — and Niu, Yue, Wu and Hu [9] study the structure of Hermitian self-dual MDS codes at length four, which is our $k=2$ row. Finally, counting *all* Hermitian self-dual codes of a given length is classical, by a mass formula; what is counted here is the MDS subfamily, which a mass formula does not reach.

Section 7 records what the same development verifies about the neighbouring existence questions, including AME$(8,4)$, where two whole routes are shut: no stabilizer code $[\![8,0,5]\!]_4$ exists at all, by Ball, Moreno and Simoens [4], and no minimal-support realisation exists either, by Bush's bound [18]. Only the non-stabilizer case is open (Remark 25), and what our development adds there is an independent machine-checked verification of special cases, not a new theorem.

2. Preliminaries

Throughout, q is a prime power, $\mathbb{F}_{q^2}/\mathbb{F}_q$ is the quadratic extension, and $\overline{x} := x^q$ is the nontrivial element of its Galois group. The norm is $\mathrm{N}(x) = x\overline{x} \in \mathbb{F}_q$; it is surjective onto $\mathbb{F}_q$, and

$$\mu := \{x \in \mathbb{F}_{q^2} : \mathrm{N}(x) = 1\}$$

is the norm-one subgroup, cyclic of order $q+1$. Every nonzero norm class $\mathrm{N}^{-1}(c)$, $c \in \mathbb{F}_q^*$, is a coset of μ and has exactly $q+1$ elements.

For $u, v \in \mathbb{F}_{q^2}^n$ put $\langle u, v\rangle_h = \sum_i u_i \overline{v_i}$, and for a linear code $C \subseteq \mathbb{F}_{q^2}^n$ let $C^{\perp_h} = \{v : \langle u, v\rangle_h = 0 \text{ for all } u \in C\}$. A code is *Hermitian self-dual* if $C = C^{\perp_h}$, which forces $n = 2k$ and $\dim C = k$; it is *MDS* if its minimum distance is $n - \dim C + 1 = k+1$. We write

$$\mathcal{H}(q,k) := \{\, C \subseteq \mathbb{F}_{q^2}^{2k} \;:\; C \text{ Hermitian self-dual MDS } [2k, k, k+1]_{q^2} \,\},$$

and $h(q,k) = |\mathcal{H}(q,k)|$.

A matrix A over $\mathbb{F}_{q^2}$ is *superregular* if every square submatrix of A is nonsingular; for a $k \times k$ matrix this is $\binom{2k}{k} - 1$ conditions, one per pair of equally sized nonempty row and column sets. Since an MDS code has every k coordinates as an information set, a code in $\mathcal{H}(q,k)$ has a unique generator matrix of the form $[\, I_k \mid A \,]$, and the following is the criterion of [1, Criterion 1, Eq. (2)].

Proposition 1 (Bevins–Bidav)**.** *The row space of* $[\, I_k \mid A \,]$ *lies in* $\mathcal{H}(q,k)$ *if and only if*

$$A\,\overline{A}^{\mathsf{T}} = -I_k \qquad \text{and} \qquad A \text{ is superregular.}$$

The map $C \mapsto A$ *is a bijection from* $\mathcal{H}(q,k)$ *onto the set of such matrices.*

Counting codes is therefore counting matrices, and this is what the searches below do. We call a matrix satisfying the two conditions of Proposition 1 an *admissible* $k \times k$ matrix and write $\mathcal{A}(q,k)$ for the set of them, so $h(q,k) = |\mathcal{A}(q,k)|$.

Definition 2. The monomial group of length n is $M_n = \mathrm{S}_n \ltimes (\mathbb{F}_{q^2}^*)^n$, acting on $\mathbb{F}_{q^2}^n$ by permuting coordinates and rescaling them. Two codes are *monomially equivalent* if some element of M_n carries one onto the other; the *class* of $C \in \mathcal{H}(q,k)$ is the set of codes in $\mathcal{H}(q,k)$ monomially equivalent to it. We write $G_k = \mathrm{S}_{2k} \ltimes \mu^{2k} \le M_{2k}$ for the subgroup with norm-one scalars, and $\mathrm{Aut}(C)$ for the stabilizer of C in G_k.

Only G_k preserves $\mathcal{H}(q,k)$, and Lemma 12 below shows that on the codes considered here the class of C is exactly its G_k-orbit, so no generality is lost by working inside G_k. Note that $\mathrm{Aut}(C)$ contains the $q+1$ norm-one global scalars, which act trivially; the stabilizer of C in the full monomial group is larger by the factor $q-1$, for the same reason.

The field model. All computations are carried out in the model $\mathbb{F}_{q^2} = \mathbb{F}_q[X]/(X^2 + c_1X + c_0)$ with q prime, an element $a + bX$ being encoded by the integer $a + qb$. For $q = 5$ we take $X^2 + X + 2$, the polynomial used in [1], so that the matrices of that paper can be read digit for digit. Irreducibility, the ring axioms on all triples, invertibility of every nonzero element, and the fact that $x \mapsto \overline{x}$ is an additive and multiplicative involution with fixed set exactly $\mathbb{F}_q$ are verified by exhaustion over the whole field for each modulus used, so nothing about the model is assumed.

3. The counting reduction

Let $D = \mathrm{diag}(d_1, \dots, d_k)$ and $E = \mathrm{diag}(e_1, \dots, e_k)$ with all $d_i, e_j \in \mu$.

Lemma 3. $A \mapsto DAE$ *maps* $\mathcal{A}(q,k)$ *to itself, and the resulting action of* $\mu^k \times \mu^k$ *has kernel exactly* $\{(cI, c^{-1}I) : c \in \mu\}$*; the induced action of the quotient, of order* $(q+1)^{2k-1}$*, is free.*

Proof. Since $E\overline{E}^{\mathsf{T}} = \mathrm{diag}(\mathrm{N}(e_j)) = I$ and likewise for D, we get $(DAE)\overline{(DAE)}^{\mathsf{T}} = D\,(A\overline{A}^{\mathsf{T}})\,\overline{D}^{\mathsf{T}} = -I$. For a submatrix on rows R and columns C,

$$\det(DAE)[R,C] = \Big(\prod_{i \in R} d_i\Big)\Big(\prod_{j \in C} e_j\Big)\det A[R,C],$$

so no minor changes between zero and nonzero and superregularity is preserved. An admissible A has no zero entry, so $DAE = A$ forces $d_i e_j = 1$ for all i, j; hence all d_i agree and all e_j agree, and $(D,E) = (cI, c^{-1}I)$. □

A *transversal* $T(q,k) \subseteq \mathcal{A}(q,k)$ for this action is obtained by normalising: scale each column j so that A_{0j} becomes the least element of its norm class, then scale each row $i \ge 1$ so that A_{i0} becomes the least element of its norm class. Because

scaling a nonzero x by μ sweeps out precisely $\mathrm{N}^{-1}(\mathrm{N}(x))$, each of these scalars exists and is unique, and the normalisation is a genuine transversal. Consequently

$$h(q,k) = |T(q,k)| \cdot (q+1)^{2k-1}. \tag{1}$$

The derivation above is by hand; its finite ingredients are verified over the whole field for every modulus used, and the multiplier (1) is in addition checked *directly* in five cells by re-running the enumeration with the normalisation switched off altogether.

Proposition 4. *For $q \in \{2,3,5,7,11,13\}$: the norm-one subgroup of $\mathbb{F}_{q^2}^*$ has order $q+1$, the norm of a nonzero element is a nonzero element of $\mathbb{F}_q$, every nonzero norm class has exactly $q+1$ elements, scaling a nonzero element by μ sweeps out exactly its norm class, and: whenever $d, d', e, e' \in \mu$ satisfy $de = d'e = de' = d'e' = 1$ then $d = d'$ and $e = e'$ — which, applied to any two rows and any two columns, is the whole content of the freeness in Lemma 3. Moreover, for $q = 5$ and $k = 6$ the number of admissible first rows up to column scaling is $819 = (4^6 - 1)/5$, the number of tuples in $(\mathbb{F}_5^*)^6$ summing to -1.*

The search itself proceeds row by row. Row 0 ranges over one representative per admissible norm profile — a tuple in $(\mathbb{F}_q^*)^k$ summing to -1, which is what $\langle r_0, r_0\rangle_h = \sum_j \mathrm{N}(r_{0j}) = -1$ says — and rows $1, \ldots, k-1$ range over the isotropic rows with no zero entry whose leading entry is a norm-class representative, filtered at each level by Hermitian orthogonality to the rows already chosen. At each level only the minors whose row set contains the new row are tested; the levels compose, so after the last row every square minor has been tested exactly once. Nothing rests on that last argument: every matrix the search returns is re-checked against the full superregularity and self-duality predicates. Both generators of rows are themselves checked against a blind filter over all $(q^2)^k$ rows in the cells where that is affordable.

4. Exact counts

Theorem 5. *Let $h(q,k)$ be the number of Hermitian self-dual MDS $[2k, k, k+1]_{q^2}$ codes. For $q \in \{2,3,5,7,11,13\}$,*

$$h(q,2) = (q-2)(q+1)^3,$$

that is 0, 64, 648, 2560, 15552, 30184 *respectively, and*

$$h(q,3) = 486,\ 12288,\ 1601856,\ 37748736,\ 2369875968,\ 10768312128$$

respectively. Furthermore

$$h(3,4) = 0, \qquad h(5,4) = 30\,569\,011\,200, \qquad h(3,5) = 2\,642\,411\,520.$$

The corresponding numbers of transversal representatives are, in each of these cells,

$$|T(q,2)| = q-2, \qquad |T(q,3)| = 2,\ 12,\ 206,\ 1152,\ 9524,\ 20022,$$

$$|T(3,4)| = 0, \qquad |T(5,4)| = 109\,200, \qquad |T(3,5)| = 10\,080.$$

Corollary 6. *$h(4,2) = 250$, $h(4,3) = 187\,500$ and $h(4,4) = 0$.*

TABLE 1. $h(q,k)$, the number of Hermitian self-dual MDS $[2k,k,k+1]_{q^2}$ codes. Blank entries are not certified here; "parked" is the $[12,6,7]_{25}$ cell of Section 6, which is nonempty but not enumerated. The three entries at $q=4$ are Corollary 6; the entry $h(2,4)=0$ is Proposition 23, there being no superregular 4×4 matrix over $\mathbb{F}_4$ at all; the other fifteen are Theorem 5.

$k\backslash q$	2	3	4	5	7	11	13
2	0	64	250	648	2560	15552	30184
3	486	12288	187500	1601856	37748736	2369875968	10768312128
4	0	0	0	30569011200			
5		2642411520					
6				parked			

Proof. Over $\mathbb{F}_{16}=\mathbb{F}_2[X]/(X^4+X+1)$ — a separate model of the field, in which μ has order 5 — the row-by-row enumeration with row 0 normalised to norm-class representatives and the remaining rows unnormalised returns 10, 1500 and 0 matrices at $k=2,3,4$. Scaling the columns by μ^k maps $\mathcal{A}(4,k)$ to itself and acts freely, since an admissible A has no zero entry, so $AE=A$ forces $E=I$; and the normalisation of row 0 is a transversal for that action, because μ-scaling sweeps out exactly a norm class. Hence $h(4,k)$ is the returned number times 5^k: $10\cdot 5^2=250$, $1500\cdot 5^3=187\,500$ and 0. The case $k=4$ is Proposition 24. □

Theorem 5 covers fifteen cells, Corollary 6 three more, and Proposition 23 the remaining entry $h(2,4)=0$ of Table 1: nineteen in all.

Two features of Table 1 are worth isolating.

Proposition 7. *There is no Hermitian self-dual MDS $[8,4,5]_9$ code, and the emptiness is genuinely the conjunction of the two conditions of Proposition 1: the 4×4 Cauchy matrix over $\mathbb{F}_9$ is superregular but not Hermitian self-dual, and a scalar matrix cI_4 with $\mathrm{N}(c)=-1$ is Hermitian self-dual but not superregular. The search dies early: five first rows survive, thirty two-row prefixes survive, and not one of them extends to three rows.*

Remark 8. Consequently the construction of [1] cannot reach AME(8, 3). That cell is in any case excluded for arbitrary states by the shadow bound of [10], whose list of nonexistent three-level AME states begins at $n=8$. The emptiness is also implicit in Danielsen's exhaustive classification of self-dual additive codes over GF(9): in his tables of the number of such codes by length and minimum distance, the largest minimum distance occurring at length 8 is 4 [5, Table 3], [6, Tables II and III]. A Hermitian self-dual $\mathbb{F}_9$-linear $[8,4,5]_9$ code would be, in particular, a trace-Hermitian self-dual additive $(8,3^8,5)$ code over GF(9), and the tables have no such entry. What is proved here is the classical statement itself, by a direct and independent enumeration.

Remark 9. $h(3,4)=0$ while $h(3,5)=2\,642\,411\,520$: over $\mathbb{F}_9$ the cell of length 8 is empty and the *longer* cell of length 10 is not. No monotonicity in k survives.

Remark 10. At $k=2$, row 0 must satisfy $\mathrm{N}(a_{00})+\mathrm{N}(a_{01})=-1$ with both norms nonzero, which has $q-2$ solutions in norm profiles, and in each of the six cells computed each profile extends to exactly one transversal representative — whence

$(q-2)(q+1)^3$. That the extension is unique for *every* q is not proved here; Theorem 5 asserts the closed form only at the six values listed.

Proof sketch of Theorem 5. The reduction to (1) is the hand argument of Section 3; its finite ingredients are Proposition 4, each verified by exhaustion over the whole field. Every value of $|T(q,k)|$ rests on an exhaustive enumeration of the transversal, run inside the proof assistant: the search tree is the row-by-row tree described in Section 3, and it is complete because row 0 ranges over a full set of column-scaling representatives and each later row over a full set of row-scaling representatives of the isotropic rows. Three independent safeguards are applied. (i) *Soundness*: every matrix returned is re-checked against the full superregularity and Hermitian self-duality predicates rather than against the incremental tests the search prunes with; this is done for the $[6,3,4]_{q^2}$ cells at $q = 3, 5, 7$ and for both large cells $[8,4,5]_{25}$ and $[10,5,6]_9$. (ii) *The multiplier*: in five cells, namely $(q,k) = (2,3)$, $(3,2)$, $(3,3)$, $(5,2)$, $(7,2)$, the entire enumeration is re-run with no normalisation anywhere — every row, including row 0, ranging over all isotropic rows without zero entries — and returns exactly the number predicted by (1). (iii) *Independent implementation*: all values were reproduced by three C programs written against different internal representations (a memoised Plücker expansion, a plain Laplace expansion, and a blind row filter), agreeing on every cell tested.

The two largest enumerations are $|T(5,4)| = 109\,200$ and $|T(3,5)| = 10\,080$; they cost, in the interpreter used for these checks, about 10 and 6 minutes respectively. The largest of the remaining ones is $|T(13,3)| = 20\,022$.

Remark 11 (computations outside the formal development). For the prime-power moduli $q = 8, 9, 16$, which the field model used here does not cover, and for two further cells too expensive to certify, the same enumeration was run only in C. It returns $h(8,2) = 4374$, $h(9,2) = 7000$, $h(16,2) = 68\,782$, $h(8,3) = 129\,435\,408$, $h(9,3) = 381\,000\,000$, $h(16,3) = 70\,231\,806\,648$, and $h(7,4) = 17\,422\,551\,613\,440$, $h(4,5) = 2\,870\,437\,500\,000$. These numbers are not machine-checked and are recorded here only for orientation; nothing below uses them.

5. Classification up to monomial equivalence

We first record why the norm-one subgroup suffices.

Lemma 12. *Let $C \in \mathcal{H}(q,k)$ and write $S(C) \subseteq \mathbb{F}_{q^2}^{2k}$ for the span of the coordinatewise products $x \circ \overline{y}$ with $x, y \in C$. If $\dim S(C) = 2k-1$, then every monomial map carrying C to a Hermitian self-dual code can be realised with all scalars in μ. In particular, monomial equivalence and G_k-equivalence agree on the codes of any cell in which this rank condition holds.*

Proof. Let $g = P \cdot \mathrm{diag}(d_1, \dots, d_{2k})$ with $(gx)_i = d_i x_{\pi(i)}$ carry C to a Hermitian self-dual C', and put $w_j = \mathrm{N}(d_{\pi^{-1}(j)})$. For $x, y \in C$,

$$0 = \langle gx, gy \rangle_h = \sum_i \mathrm{N}(d_i)\, x_{\pi(i)} \overline{y_{\pi(i)}} = \sum_j w_j\, x_j \overline{y_j},$$

so w annihilates $S(C)$ under the standard bilinear pairing. The all-ones vector annihilates $S(C)$ as well, since C is Hermitian self-dual. If $\dim S(C) = 2k-1$ the annihilator is a line, so $w = c \cdot (1, \dots, 1)$ for some c, necessarily in $\mathbb{F}_q^*$; choosing e with $\mathrm{N}(e) = c$ makes every d_i/e norm-one, and the global scalar e acts trivially on a linear code. □

The rank $\dim S(C)$ is a monomial invariant, so the hypothesis of Lemma 12 needs to be verified only once per class; it is verified below at a representative of every class of every cell classified, and in the cheap cells at *every* transversal representative.

Theorem 13. *The numbers of monomial equivalence classes of Hermitian self-dual MDS $[2k, k, k+1]_{q^2}$ codes are as follows.*

cell	*codes*	*classes*	*class sizes (as normalised representatives)*
$[4,2,3]_9$	64	1	1
$[4,2,3]_{25}$	648	1	3
$[4,2,3]_{49}$	2560	2	2, 3
$[4,2,3]_{121}$	15552	2	3, 6
$[4,2,3]_{169}$	30184	3	2, 3, 6
$[6,3,4]_4$	486	1	2
$[6,3,4]_9$	12288	1	12
$[6,3,4]_{25}$	1601856	4	6, 20, 60, 120
$[6,3,4]_{49}$	37748736	8	12, 60, 60, 120, 120, 180, 240, 360
$[6,3,4]_{121}$	2369875968	30	
$[6,3,4]_{169}$	10768312128	51	
$[8,4,5]_{25}$	30569011200	7	3360, 5040, 6720, 13440, 20160, 20160, 40320
$[10,5,6]_9$	2642411520	1	10080

Here a "class size" is the number of transversal representatives in the class, so the number of codes in it is that number times $(q+1)^{2k-1}$; the class sizes of a cell sum to $|T(q,k)|$. The full lists of class sizes for $[6,3,4]_{121}$ and $[6,3,4]_{169}$ are omitted here for space and are part of the verified statements.

Corollary 14. *A code in a class of normalised size s has $|\mathrm{Aut}(C)| = (2k)!\,(q+1)/s$ inside $G_k = \mathrm{S}_{2k} \ltimes \mu^{2k}$. In particular the seven classes of the* $\mathrm{AME}(8,5)$ *cell $[8,4,5]_{25}$ have automorphism groups of orders $72, 48, 36, 18, 12, 12, 6$; the unique class of the* $\mathrm{AME}(10,3)$ *cell $[10,5,6]_9$ has $|\mathrm{Aut}| = 1440$; and the unique class of $[6,3,4]_4$ has $|\mathrm{Aut}| = 1080$.*

Proof. Orbit–stabilizer for G_k acting on $\mathcal{H}(q,k)$, together with (1): the orbit of C has $s\,(q+1)^{2k-1}$ elements and $|G_k| = (2k)!\,(q+1)^{2k}$. □

Proof sketch of Theorem 13. Fix a cell and let $T = T(q,k)$ be its transversal, computed as in Section 4. Working modulo the diagonal action means working on T, and on T the group G_k acts through the following three kinds of move, applied and then re-normalised:

- adjacent row swaps of A and adjacent column swaps of A, generating $\mathrm{S}_k \times \mathrm{S}_k$;
- the *pivot* at position $(0,0)$, which is the effect on A of transposing the code coordinates 0 and k and re-normalising $[\,I_k \mid A\,]$, namely

$$A_{00} \mapsto A_{00}^{-1}, \quad A_{0j} \mapsto A_{0j}A_{00}^{-1}, \quad A_{i0} \mapsto -A_{i0}A_{00}^{-1}, \quad A_{ij} \mapsto A_{ij} - A_{i0}A_{0j}A_{00}^{-1}$$

  (well defined since $A_{00} \neq 0$).

Adjacent transpositions generate each S_k, and one transposition across the two halves conjugated by them gives all of S_{2k}, so these moves generate the full coordinate group over the diagonal action. The classification is then the connected component decomposition of T under the moves, computed by breadth-first search. The computation carries its own soundness check: every move of every element of

T is required to normalise back *into* T, and a single failed lookup would flip the returned closure flag, which is part of each verified statement. Where independent representatives were available they are supplied to the same computation and certified to be solutions lying in pairwise distinct components, and two negative controls are run — a deliberately duplicated representative and a matrix that is not a solution are both rejected. Finally Lemma 12 is applied: the conjugated-product rank $2k-1$ is verified at one representative of each class for the $k \geq 3$ cells, and at *every* element of T for all five $k = 2$ cells and for the two $k = 3$ cells over $\mathbb{F}_{121}$ and $\mathbb{F}_{169}$, so all thirteen tables are statements about full monomial equivalence. The two largest closures, $[8, 4, 5]_{25}$ over 109 200 representatives and $[10, 5, 6]_9$ over 10 080, were each run as a single certified computation. All component decompositions were reproduced by an independent C implementation.

Remark 15. Three rows of the table recover published facts and serve as controls rather than as results. $[6, 3, 4]_4$ is the hexacode cell: a single class, and $|\mathrm{Aut}| = 1080 = |3 \cdot A_6|$, the classical automorphism group of the hexacode. $[6, 3, 4]_9$ and $[10, 5, 6]_9$ have a single class each, agreeing with Danielsen's exhaustive classifications of self-dual *additive* codes over GF(9): his tables give exactly one $(6, 3^6, 4)$ code and exactly one $(10, 3^{10}, 6)$ code [5, Table 3], [6, Tables II and III], and [6] records of that earlier work that these two, with $(4, 3^4, 3)$, are the only nontrivial self-dual additive MDS codes over GF(9) if the MDS conjecture holds. For the $(10, 3^{10}, 6)$ code [6] records "a unique extremal $(10, 3^{10}, 6)$ code ... and automorphism group of size 2880" — exactly twice our monomial 1440. The two categories are not the same — an additive code over GF(9) is only $\mathbb{F}_3$-linear, and its equivalence group is $\mathrm{Sp}_2(3) \wr \mathrm{S}_n$ [6, Sec. I], of order 24 per coordinate against our $q+1 = 4$ — so the factor 2 is a consistency check and not an identity of statements. The remaining rows — $[8, 4, 5]_{25}$ and the $[6, 3, 4]_{q^2}$ cells for $q = 7, 11, 13$ — we could not locate in the literature; [5] classifies self-dual additive codes over GF(25) only up to length 6, and over GF(q^2) not at all for $q \geq 7$.

6. The class of the Bevins–Bidav $[12, 6, 7]_{25}$ code

Let $\mathbb{F}_{25} = \mathbb{F}_5(\alpha)$ with $\alpha^2 + \alpha + 2 = 0$ and encode $a + b\alpha$ by $a + 5b$, as in [1]. The code constructed there is the row space of $[\, I_6 \mid A \,]$ with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 23 & 1 & 2 & 3 & 5 & 3 \\ 19 & 7 & 1 & 22 & 17 & 12 \\ 9 & 15 & 11 & 2 & 18 & 15 \\ 1 & 4 & 22 & 9 & 2 & 1 \\ 6 & 7 & 18 & 15 & 10 & 2 \end{pmatrix}, \tag{2}$$

whose $923 = \binom{12}{6} - 1$ square minors are all nonzero and which satisfies $A\overline{A}^{\mathsf{T}} = -I_6$. Call this code C_A. The full enumeration of the cell $[12, 6, 7]_{25}$ is out of reach (Remark 20); the equivalence question about the code itself is not.

Theorem 16. *Let C_A be the code of* (2) *and $\overline{C_A}$ its entrywise Galois image. Then:*

1. $|\mathrm{Aut}(C_A)| = 108$ *inside* $G_6 = \mathrm{S}_{12} \ltimes \mu^{12}$*, and the image of* $\mathrm{Aut}(C_A)$ *in* S_{12} *has order* 18*;*

(2) *the monomial equivalence class of* C_A *contains exactly*

$$\frac{12! \cdot 6^{11}}{18} = 9\,654\,465\,016\,627\,200$$

Hermitian self-dual MDS $[12, 6, 7]_{25}$ *codes;*

(3) $\overline{C_A}$ *is again a Hermitian self-dual MDS* $[12, 6, 7]_{25}$ *code and is* not *monomially equivalent to* C_A*.*

In particular the cell $[12, 6, 7]_{25}$ *contains at least* $2 \cdot 9\,654\,465\,016\,627\,200$ *codes in at least two monomial equivalence classes.*

Proof sketch. Two finite objects are enumerated. First, the *block orbit*: the $(6!)^2 = 518\,400$ matrices PAQ with P, Q permutation matrices, each normalised as in Section 3, are pairwise distinct. Hence the stabilizer of C_A inside the split-preserving subgroup $(\mathrm{S}_6 \times \mathrm{S}_6) \ltimes \mu^{12}$, modulo the global scalars, is trivial; write $s = 1$. Second, the *split orbit*: since C_A is MDS, each of the $\binom{12}{6} = 924$ six-element sets S of coordinates is an information set, and Gaussian elimination of the generator matrix with the columns of S first produces the matrix A_S presenting the same code at that split. The set S lies in the $\mathrm{Aut}(C_A)$-orbit of the standard split exactly when the normalisation of A_S lies in the block orbit above; this happens for exactly $m = 18$ of the 924 splits. (That every split does eliminate fully is verified as well, an independent rederivation of the MDS property through Gaussian elimination rather than through minors.) Orbit–stabilizer now gives $|\mathrm{Aut}(C_A)| = (q+1)\, s\, m = 6 \cdot 1 \cdot 18 = 108$, the factor $q + 1$ being the global scalars, which act trivially, and

$$\#\{\text{codes in the class}\} = \frac{|G_6|}{|\mathrm{Aut}(C_A)|} = \frac{12! \cdot 6^{12}}{108} = \frac{12! \cdot 6^{11}}{18}.$$

Lemma 12 applies because the conjugated products of C_A span a space of dimension exactly $2k - 1 = 11$, verified by an exact rank computation, so the class is the full monomial class and not merely the G_6-class. For (3), $\overline{A}$ satisfies both conditions of Proposition 1, and running the same split test for $\overline{A}$ against the block orbit of A returns 0 of 924: no monomial map carries one code to the other. All the arithmetic in this proof — the 518 400 normalisations, the 2×924 Gaussian eliminations, the two rank computations — is exhaustive and machine-checked; only the orbit–stabilizer bookkeeping is by hand. □

Remark 17 (the membership test is not vacuous). A certificate of the form "0 of 924" is only as good as the search that returns 0. The same test returns 18 on C_A itself, finds the standard split, and recovers every one of the 518 400 stored orbit keys by the same binary search; on the three smaller witnesses of Section 7 it returns 20, 10 and 8. An independent C implementation reproduces $s = 1$, $m = 18$ and the class size, and additionally computes the breadth-first closure of the normalisation of A under the row/column swaps and the pivot of Section 5: it visits exactly $26\,611\,200 = 12!/(s \cdot m)$ normalised matrices, which is the number predicted by the bookkeeping above, along a code path sharing nothing with the split enumeration.

Proposition 18. *The same machinery applied to the three smaller codes of Section 7 returns, with* $s = (k!)^2/(\text{block orbit size})$ *as in the proof of Theorem 16: for the hexacode* $[6, 3, 4]_4$*, block orbit* 2 *(so* $s = 18$*) and* $m = 20$*, hence class size* $6! \cdot 3^5/(18 \cdot 20) = 486$ *— the whole cell — and* $|\mathrm{Aut}| = 1080$*; for the* $[6, 3, 4]_9$ *code, block orbit* 6 *(so* $s = 6$*) and* $m = 10$*, hence class size* $6! \cdot 4^5/(6 \cdot 10) = 12\,288$*, again*

the whole cell; for the $[8,4,5]_{25}$ *code, block orbit* 576 *(so* $s=1$*) and* $m=8$*, hence class size* $8!\cdot 6^7/8 = 1\,410\,877\,440 = 5040\cdot 6^7$*, which is the class of normalised size* 5040 *in Theorem 13. In addition, the Schur square of the code* (2) *has dimension* $\dim C_A^{\star 2} = 12$*, and* $\dim S(C_A) = 11$*; the same two values hold for* $\overline{C_A}$*.*

The value $\dim C_A^{\star 2} = 12$ is the one structural invariant [1] computes for its code, from which that paper concludes that the code is not monomially equivalent to a generalised Reed–Solomon code, a $[12,6]$ GRS code having Schur square of dimension at most 11. We certify the rank. The GRS bound is used nowhere below and is not re-proved here; it is in any case elementary, since the coordinatewise square of a generalised Reed–Solomon $[n,k]$ code with evaluation points a and multipliers v lies in the generalised Reed–Solomon code with points a and multipliers v^2 of dimension $\min(n, 2k-1)$. ([1] attaches a citation to [2] at this point; that paper is about squares of *random* linear codes and does not state the GRS bound, so we record the bound as standard rather than attributing it.)

Remark 19. Theorem 16(1) is consistent with the only automorphism statement in [1], which reports that "a subsequent automorphism calculation revealed a regular $H \cong \mathbb{Z}_3^2$ orbit on nine coordinates": the image of $\mathrm{Aut}(C_A)$ in S_{12} has order 18, which is divisible by $|\mathbb{Z}_3^2| = 9$.

Remark 20 (the cell itself is out of reach). A complete enumeration of $[12,6,7]_{25}$ is not affordable. The transversal search has 819 first-row classes and 105 814 800 two-row prefixes, measured; extrapolating the measured cost of one first-row class gives roughly 9×10^{10} three-row prefixes and on the order of 130 core-hours for the third level alone, with three further levels above it. A canonical augmentation scheme would be the right instrument if this is ever wanted. We also transplanted the group-matrix ansatz of [1] — which produced their $[18,9,10]$ codes — to $k=6$: over both groups of order 6, all $25^6 = 244\,140\,625$ kernels give 46 656 respectively 25 920 Hermitian group matrices and not one is superregular. (That last computation was run only in C and is not machine-checked; it explains why no cheap third class turned up here.)

7. Verified background

The results above sit on top of a layer of certificates for the codes themselves, which we record for completeness; none of it is claimed as new.

Proposition 21. *The three matrices printed in* [1] *do what that paper says. The* 6×6 *matrix* (2) *over* $\mathbb{F}_{25}$ *satisfies* $A\overline{A}^{\mathsf{T}} = -I_6$ *and has all* 923 *square minors nonzero, so it generates a Hermitian self-dual MDS* $[12,6,7]_{25}$ *code. The* 9×9 *matrices over* $\mathbb{F}_{121}$ *and over* $\mathbb{F}_{169}$*, rebuilt from the* 3×3 *group-circulant kernels of* [1] *by that paper's own reconstruction and its convolution identity, satisfy the same two conditions with all* $48\,619 = \binom{18}{9} - 1$ *minors nonzero, so they generate Hermitian self-dual MDS* $[18,9,10]_{121}$ *and* $[18,9,10]_{169}$ *codes.*

Proposition 22. *Hermitian self-dual MDS codes* $[6,3,4]_4$ *(the hexacode),* $[6,3,4]_9$ *and* $[8,4,5]_{25}$ *exist, with explicit generator matrices; there are exactly* 486 *superregular* 3×3 *matrices over* $\mathbb{F}_4$*, and the hexacode block is one of them. There is no superregular* 3×3 *matrix over* $\mathbb{F}_3$*, hence no linear* $[6,3,4]_3$ *MDS code and no minimal-support* $\mathrm{AME}(6,3)$ *state with a linear defining code — even though*

$\mathrm{AME}(6,3)$ *exists, by the* $[6,3,4]_9$ *code. The two routes are therefore genuinely different, and this is the cheapest place to see it.*

Proposition 23. *There is no superregular* 4×4 *matrix over* $\mathbb{F}_4$*: exhaustively, none of the* $81\times256=20\,736$ *two-row extensions of a superregular first row survives. Hence there is no linear* $[8,4,5]_4$ *MDS code.*

Proposition 24. *There is no* $\mathbb{F}_{16}$*-linear Hermitian self-dual MDS* $[8,4,5]_{16}$ *code. Exhaustively up to the monomial reduction of Section 3: from* 20 *normalised first rows the search finds* 9600 *two-row prefixes and not one extends to three rows; the reduction's finite content (the norm-one subgroup of* $\mathbb{F}_{16}$ *has order* 5*, and norm-one scaling sweeps out exactly a norm class) is verified separately, and the whole search was reproduced in C in a different presentation of* $\mathbb{F}_{16}$*. The positive controls at* $k=2$ *and* $k=3$ *return the corresponding* $\mathrm{AME}(4,4)$ *and* $\mathrm{AME}(6,4)$ *codes.*

Remark 25. Propositions 23 and 24 concern $\mathrm{AME}(8,4)$, of which [3] writes "the first remaining open case is the existence of an $\mathrm{AME}(8,4)$ state". Neither is new, and neither bears on what is still open there. Ball, Moreno and Simoens [4] proved that no stabilizer code $[\![8,0,5]\!]_4$ exists at all, which closes the entire stabilizer class, and Bush's bound for orthogonal arrays of index unity [18], [19, Thm. 2.19] — in the weak form $n\le q+k-1$, which at $q=4$, $k=4$ reads $n\le7$ — already excludes any $(8,256,5)_4$ MDS code, linear or not, hence any minimal-support realisation. What remains open is the non-stabilizer case. What the present development contributes on that cell is independent machine-checked verification of special cases of published theorems.

8. Verification

Every numerical claim above — every count, every class number and class size, every automorphism order, every rank, and every emptiness — has been formalised and checked in Lean 4, and the checks are reproducible from the accompanying development; repository reference to be added at submission. The exceptions, proved by hand and marked as such where they occur, are Proposition 1 (quoted from [1]), Lemma 3, Lemma 12, and the orbit–stabilizer bookkeeping of Corollary 14 and Theorem 16; each of these consumes only finite inputs that are themselves checked. The development defines $\mathbb{F}_{q^2}$, determinants by Laplace expansion with explicit recursion depth, the minor list, the Hermitian form and the two predicates of Proposition 1 directly on lists of natural numbers, imports no external library for the enumerations (so that the kernel can reduce them), and verifies the field model itself over its whole domain rather than assuming it. The enumerations are discharged either by kernel reduction (`decide +kernel`) or, where their size makes that impractical, by compiled evaluation (`native_decide`), which adds one evaluation axiom per declaration; an audit of the axiom set of all 110 statements involved in Sections 3–6 found no other axioms than `propext`, `Quot.sound` and those evaluation axioms, and in particular no incomplete proof anywhere. Nineteen of the 110 use no axiom at all, and the rank computations at the class representatives that feed Lemma 12 use only propositional extensionality, so the finite input to the norm-one reduction is free of any evaluation axiom at every class of every cell classified. (The two sweeps that verify the rank at *every* element of a transversal — the five $k=2$ cells and the two $k=3$ cells over $\mathbb{F}_{121}$ and $\mathbb{F}_{169}$ — are stronger statements and do use the evaluation axiom.) Mathlib is used only in a companion

module, for the orthogonal-array argument mentioned in Remark 25, and by no statement of this paper. A memoised Plücker expansion is used in place of the naive Laplace expansion for the largest minor computations; it is pinned to the naive one by an entry-by-entry, sign-by-sign agreement on all 923 minors of (2), and on a deliberately corrupted variant where both routes find the same eleven vanishing minors. Negative controls are run throughout: a Hermitian self-dual matrix that is not superregular and a superregular matrix that is not Hermitian self-dual show that neither half of Proposition 1 is redundant, and corrupting a single entry of (2) breaks both halves. The three transfers we quote but do not formalise — the minimal-support/MDS correspondence [13, 14], the stabilizer construction [15], and the shortening step [14] — are cited only; no statement in the formal development mentions a quantum state.

9. Open questions

(1) *Does* $[12, 6, 7]_{25}$ *have a third class?* Theorem 16 gives two, and they are one under semilinear equivalence. Deciding whether there are others needs either the enumeration parked in Remark 20 or an ansatz; the natural candidate ansatz, the group-matrix construction of [1], is empty at $k = 6$.

(2) *Is* $h(q, 2) = (q - 2)(q + 1)^3$ *for all* q*?* The count of admissible first-row profiles is $q - 2$ in general; what is verified only case by case is that each profile extends to exactly one transversal representative.

(3) *Are the class numbers* $1, 1, 4, 8, 30, 51$ *of the* $[6, 3, 4]_{q^2}$ *cells at* $q = 2, 3, 5, 7, 11, 13$ *an instance of a formula?* We found no match; the class-count and transversal-count sequences we queried are absent from OEIS.

(4) AME$(8, 4)$. Only the non-stabilizer case is open (Remark 25); no finite search of the kind used here bears on it.

References

[1] S. Bevins and Y. Bidav, *Symmetry-guided constructions of absolutely maximally entangled states in five open cases*, arXiv:2608.05781v2 (2026). All quotations from and numbered references to this paper below are against v2 (10 August 2026), the current version.

[2] I. Cascudo, R. Cramer, D. Mirandola and G. Zémor, *Squares of random linear codes*, IEEE Trans. Inform. Theory **61** (2015), no. 3, 1159–1173; arXiv:1407.0848.

[3] F. Shi, X. Zhang, Q. Zhao and L. Li, *Complete existence classification of seven-partite absolutely maximally entangled states*, arXiv:2608.01011v1 (2026).

[4] S. Ball, E. Moreno and R. Simoens, *Stabilizer codes over fields of even order*, IEEE Trans. Inform. Theory **71** (2025), no. 5, 3707–3718; DOI 10.1109/TIT.2024.3454480. Preprint: arXiv:2401.06618 (2024), where the title is spelled *Stabiliser codes over fields of even order*.

[5] L. E. Danielsen, *Graph-based classification of self-dual additive codes over finite fields*, Adv. Math. Commun. **3** (2009), no. 4, 329–348; arXiv:0801.3773. Table numbers are those of the e-print.

[6] L. E. Danielsen, *On the classification of Hermitian self-dual additive codes over* GF(9), IEEE Trans. Inform. Theory **58** (2012), no. 8, 5500–5511; arXiv:1106.2428. Table numbers and the section number are those of the e-print.

[7] M. Grassl and M. Rötteler, *Quantum MDS codes over small fields*, Proc. 2015 IEEE Int. Symp. Inform. Theory (ISIT 2015), 1104–1108; arXiv:1502.05267.

[8] S. Bouyuklieva, *Some MDS codes over* $GF(64)$ *connected with the binary doubly-even* $[72, 36, 16]$ *code*, Serdica J. Comput. **1** (2007), 185–192. We were unable to obtain this paper; the reference is as given in the bibliography of arXiv:1711.05710.

[9] Y. Niu, Q. Yue, Y. Wu and L. Hu, *Hermitian self-dual, MDS, and generalized Reed–Solomon codes*, IEEE Commun. Lett. **23** (2019), no. 5, 781–784; DOI 10.1109/LCOMM.2019.2908640. We were unable to obtain this paper.

[10] F. Huber, C. Eltschka, J. Siewert and O. Gühne, *Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity*, J. Phys. A: Math. Theor. **51** (2018), 175301; arXiv:1708.06298.

[11] A. J. Scott, *Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions*, Phys. Rev. A **69** (2004), 052330; arXiv:quant-ph/0310137v3. Proposition 3 is numbered as in that e-print.

[12] W. Helwig, W. Cui, J. I. Latorre, A. Riera and H.-K. Lo, *Absolute maximal entanglement and quantum secret sharing*, Phys. Rev. A **86** (2012), 052335; arXiv:1204.2289. (The arXiv author list interchanges Riera and Latorre; the order above is the published one.)

[13] W. Helwig and W. Cui, *Absolutely maximally entangled states: existence and applications*, arXiv:1306.2536 (2013).

[14] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera and K. Życzkowski, *Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices*, Phys. Rev. A **92** (2015), 032316; arXiv:1506.08857.

[15] A. Ketkar, A. Klappenecker, S. Kumar and P. K. Sarvepalli, *Nonbinary stabilizer codes over finite fields*, IEEE Trans. Inform. Theory **52** (2006), no. 11, 4892–4914; arXiv:quant-ph/0508070v2. Corollary 19 is numbered as in that e-print.

[16] A. Ashikhmin and E. Knill, *Nonbinary quantum stabilizer codes*, IEEE Trans. Inform. Theory **47** (2001), no. 7, 3065–3072; arXiv:quant-ph/0005008.

[17] E. M. Rains, *Nonbinary quantum codes*, IEEE Trans. Inform. Theory **45** (1999), no. 6, 1827–1832; arXiv:quant-ph/9703048 (1997).

[18] K. A. Bush, *Orthogonal arrays of index unity*, Ann. Math. Statist. **23** (1952), no. 3, 426–434; DOI 10.1214/aoms/1177729387.

[19] A. S. Hedayat, N. J. A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer Series in Statistics, Springer, New York, 1999.

[20] F. Huber and N. Wyderka, *Table of AME states*, online at `https://huberfe.github.io/ame` (accessed 2026-08-30; the older address `http://www.tp.nt.uni-siegen.de/+fhuber/ame.html` redirects there). The cell data quoted here is the Internet Archive snapshot of 2023-02-01, `https://web.archive.org/web/20230201113713/https://www.tp.nt.uni-siegen.de/+fhuber/table.html`; we checked it against the current version of the table, dated 2026-06-08, and no cell used in this paper differs. The table asks that [10], where its bounds first appeared, be cited alongside it.