BOUNDED-DEGREE INDUCED FORESTS IN THE MINIMUM-ORDER COUNTEREXAMPLE TO THE ALBERTSON–BERMAN CONJECTURE

KOREA SUPERINTELLIGENCE LABS

Abstract. For a graph G let $a(G)$ be the maximum order of an induced forest of G, and for an integer $d \geq 2$ let $f_d(G)$ be the maximum order of an induced forest of maximum degree at most d. Albertson and Berman conjectured in 1979 that $a(G) \geq |V(G)|/2$ for every planar G; the conjecture was disproved in August 2026, and Cames van Batenburg, Goedgebeur and Jooken showed that the minimum order of a counterexample is 29, exhibiting two graphs G_{29} and G'_{29} of that order with $a = 14$. We prove that $f_d(G_{29}) = f_d(G'_{29}) = 14$ for every $d \geq 2$: on these two graphs the degree restriction costs nothing, because some maximum induced forest is already a union of paths. Consequently both graphs violate the conjecture $f_d(G) > 2dn/(4d+1)$ of Chappell and Pelsmajer for exactly the range $d \geq 7$, and *satisfy* it for $2 \leq d \leq 6$, so they leave the cases that are open untouched. The threshold is exact rather than an artefact: $29 = 4 \cdot 7 + 1$ and $14 = 2 \cdot 7$, so $2dn = (4d + 1)f_d$ at $d = 7$ and it is the strictness of the inequality that fails. The refutation for $d \geq 7$ already follows from the published value $a(G_{29}) = 14$, and lowers from 52 to 29 the smallest order at which a counterexample to that conjecture has been recorded; what the theorem adds is the converse half. Along the way we re-certify $a(G_{29}) = a(G'_{29}) = 14$ by a route that does not use the exhaustive gadget tables of the original argument: one hereditary density inequality together with four explicit 2-core certificates spanning 22 vertices in total. Planarity, which is the one ingredient we do not formalise, is isolated as an explicit hypothesis and supported by a spherical rotation-system certificate whose checker rejects all 46 656 rotation systems of $K_{3,3}$. Every theorem below is machine-checked in Lean 4; the few counts that are not are flagged where they occur.

1. Introduction

An *induced forest* of a graph G is a set $S \subseteq V(G)$ such that $G[S]$ contains no cycle; $a(G)$ denotes the largest order of such a set, and

$$c_{\mathcal{P}} := \inf\left\{ \frac{a(G)}{|V(G)|} \ : \ G \text{ planar} \right\}.$$

Since every forest is bipartite, an induced forest on half the vertices yields an independent set on a quarter of them, so lower bounds on $c_{\mathcal{P}}$ strengthen the classical consequence of the Four Colour Theorem. Borodin's acyclic 5-colouring theorem [2] gives $c_{\mathcal{P}} \geq 2/5$, and $a(K_4) = 2$ gives $c_{\mathcal{P}} \leq 1/2$. In 1979 Albertson and Berman [1] conjectured that the upper bound is the truth.

Conjecture 1.1 (Albertson–Berman [1], in the wording of [8])**.** Let G be a simple planar graph of order n, and let k be the size of a largest set of vertices of G which induces a forest. Then k/n is at least $1/2$; equivalently, $c_{\mathcal{P}} \geq 1/2$.

The conjecture stood for almost five decades — it appears in the open problem collections of West [9] and Mohar [8] — and was disproved in August 2026, by several authors independently. Jung [6] gave a plane triangulation T on 31 vertices with $a(T) = 15$; Makarov [7] gave a counterexample on 39 vertices with $a = 19$ and indicated, without a full proof, a 76-vertex variant with ratio 37/76. Cames van Batenburg, Goedgebeur and Jooken [3] then settled the extremal question: the minimum order of a counterexample is 29, and they exhibit two graphs of that order, a graph G_{29} with 80 edges and the planar triangulation G'_{29} obtained from it by adding one edge, both with $a = 14$, so that $c_{\mathcal{P}} \leq 14/29$. Their minimality half is expensive — it rests on generating all 16 747 182 732 792 cubic planar 3-connected graphs without triangular faces on 50 vertices, which they price at approximately 10 CPU years on a cluster — and we use it only as a citation; nothing below re-verifies it.

The same paper draws attention to a bounded-degree strengthening. For a graph G and a nonnegative integer d, Chappell and Pelsmajer [4] define $f_d(G)$ to be the maximum order of an induced forest of G having maximum degree at most d, and conjecture:

Conjecture 1.2 (Chappell–Pelsmajer [4, Conjectures 4 and 20])**.** Let $d \geq 2$ be an integer. If G is a planar graph with order $n \geq 1$, then $f_d(G) > \dfrac{2dn}{4d+1}$.

We quote it verbatim; [4] states it twice in identical words, as its Conjecture 4 and again as its Conjecture 20. The restriction $d \geq 2$ is needed: at $d = 1$ the octahedron $K_{2,2,2}$ has $f_1 = 2 < 12/5$ [4].

Since $f_d(G) \leq a(G)$, any planar graph with $a(G) \leq 2dn/(4d+1)$ refutes Conjecture 1.2 at d. Applying this to their infinite family with ratio 25/52, the authors of [3] obtain counterexamples for every $d \geq 7$; the smallest member of that family has 52 vertices. For $2 \leq d \leq 6$ they list the conjecture as open. Applying the same arithmetic to the published values of a for the other counterexamples of [6, 7, 3] gives refutations for $d \geq 8$ (ratio 15/31) and for $d \geq 10$ (ratios 19/39, 37/76 and 20/41), and nothing smaller by that route.

The minimum-order counterexample lies exactly on the boundary. Our main result is that on G_{29} and G'_{29} the degree restriction in f_d is free.

Theorem 1.3 (Theorem 7.2 and Corollary 7.3 below)**.** *For every integer $d \geq 2$,*

$$f_d(G_{29}) = f_d(G'_{29}) = 14.$$

Consequently G_{29} and G'_{29} are counterexamples to Conjecture 1.2 for every $d \geq 7$, on 29 vertices, and they satisfy Conjecture 1.2 for every $2 \leq d \leq 6$.

Three things are worth saying about this.

First, the arithmetic behind the range is one line once $f_d = 14$ is known: $2d \cdot 29 < (4d+1) \cdot 14$ holds precisely when $d < 7$. What is not a line is the input. That $f_d(G_{29}) = a(G_{29})$ for all $d \geq 2$ requires a maximum induced forest of maximum degree at most 2, and there is no reason in general for an extremal induced forest to be linear: had every maximum induced forest of G_{29} contained a vertex of large degree, $f_6(G_{29})$ could have dropped to 13 and one of the cases $2 \leq d \leq 6$ that [3] leaves open would have fallen. Theorem 1.3 says that this does not happen, and the negative half — that the minimum-order counterexample settles none of the open cases — is as much of the content as the positive half.

Second, the order drops from 52 to 29. We claim no depth for this half: it needs only the published value $a(G_{29}) = 14$, since $f_d \leq a$ and $14 \leq 2d \cdot 29/(4d+1)$ as soon as $d \geq 7$. In other words the minimum-order counterexample of [3] already refutes Conjecture 1.2 over the whole range that [3] obtains from its 52-vertex family, and 29 is the smallest order at which $d = 7$ is reached by any published counterexample to Conjecture 1.1 (Table 3). We have not found this recorded anywhere: on 2026-08-29 we searched an arXiv full-text mirror of 881 145 papers (newest identifier `2608.27451`), the arXiv metadata interface, and the nine works that OpenAlex and Semantic Scholar list as citing [4]; the only document that states Conjecture 1.2 at all is [3], which draws the consequence only for its 52-vertex family, and neither [6] nor [7] mentions bounded degree anywhere. What does need Theorem 1.3 is the converse half, and with it the exactness of the range.

Third, the threshold $d = 7$ is exact and not an artefact of rounding: $29 = 4 \cdot 7 + 1$ and $14 = 2 \cdot 7$, so at $d = 7$ the two sides of Conjecture 1.2 are equal, $2 \cdot 7 \cdot 29 = 406 = (4 \cdot 7 + 1) \cdot 14$, and what the graph violates is the strictness of the inequality (Proposition 7.4).

Sections 2–6 set this up and, in the process, certify by an independent route the input that Theorem 1.3 rests on. The value $a(G_{29}) = 14$ is due to [3] and we claim no priority for it; what we contribute there is a different and much smaller certificate. The argument of [3] splits G_{29} into the two gadgets Q and R and uses exhaustively computed tables p_Q and p_R of the best induced forests of each gadget with prescribed trace on the identified edge. We keep the split, but replace the verification of the tables by a single hereditary inequality — an induced forest on T has fewer than $|T|$ edges — applied to every subset of each gadget, together with exactly four explicit 2-core certificates that dispose of the five subsets on which the inequality alone is not enough. Those certificates are three induced 6-cycles and one induced 4-cycle, 22 vertices in total, and are small enough to print (Table 2).

Planarity is the one notion in Conjecture 1.1 that we do not formalise, and we do not pretend otherwise. Theorem 4.5 is stated relative to an arbitrary predicate: whatever "planar" means, as soon as it holds of G_{29} the conjecture fails. Section 5 then supplies the standard combinatorial certificate of that single hypothesis — a rotation system of G_{29} whose face count satisfies Euler's formula for the sphere — and guards the checker on both sides, the strongest guard being that all 46 656 rotation systems of $K_{3,3}$ are rejected. Section 6 records the icosahedron as a positive control and several negative controls. Section 8 records that G_{29} is vertex-critical, and Section 9 lists what we could not reach.

2. Preliminaries

All graphs are finite and simple. For $S \subseteq V(G)$ we write $G[S]$ for the induced subgraph, $N_S(v) = \{w \in S : vw \in E(G)\}$ and $d_S(v) = |N_S(v)|$. We abbreviate the degree sum of an induced subgraph by

$$\sigma_G(S) \ := \ \sum_{v \in S} d_S(v) \ = \ 2\,\big|E(G[S])\big|.$$

A set S is an *induced forest* if $G[S]$ is acyclic, and $a(G)$ is the largest order of one. For $d \geq 2$, $f_d(G)$ is the largest order of an induced forest S with $d_S(v) \leq d$ for all $v \in S$. Thus $f_2(G) \leq f_3(G) \leq \cdots \leq a(G)$, and an induced forest with $d_S(v) \leq 2$ throughout is a disjoint union of paths, which we call a *linear* forest. Finally $\alpha(G)$ is the independence number.

The two graphs. Following [3], let Q be the 14-vertex gadget of their Figure 1 with distinguished edge xy, and R the 17-vertex gadget of their Figure 2 with distinguished edge bc; G_{29} is obtained by identifying xy with bc, so it has $14 + 17 - 2 = 29$ vertices and $36 + 45 - 1 = 80 = 3 \cdot 29 - 7$ edges. Both gadgets are icosahedron modifications: deleting x, y from Q and adding the edge between the two common neighbours of x and y gives the icosahedron, and contracting the subgraph induced by the closed neighbourhood of $\{b, c\}$ in R to a single vertex gives the icosahedron as well [3]. The graph G'_{29} is G_{29} plus one further edge; up to isomorphism that is the only way to complete G_{29} to a planar triangulation [3], and G'_{29} has $81 = 3 \cdot 29 - 6$ edges.

We fix once and for all the vertex set $V = \{0, 1, \ldots, 28\}$ and the adjacency of Table 1; G'_{29} is G_{29} together with the edge 16 20. This labelling is read off the drawing of Figure 3 of [3]. In `graph6` format the two graphs are

G_{29} `\JYCI_oQPPO____@_o????S??g?OoWb?W??O??C?GH@?OSA?@SI??sD_?CF_??z_??BkK`
G'_{29} `\JYCI_oQPPO____@_o????S??g?OoWb?W??W??C?GH@?OSA?@SI??sD_?CF_??z_??BkK`

(computed from Table 1 and decoded back to check the round trip; the two strings differ in one character). The first of them is, character for character, the `graph6` string that the House of Graphs [5] stores for its entry `57246`, "A 29-vertex counterexample to Albertson-Berman that is edge-critical", on 29 vertices with 80 edges: our labelling is not merely isomorphic to the deposited graph but equal to it. For G'_{29} the corresponding entry is `57245`, "A 29-vertex counterexample to Albertson-Berman", on 29 vertices with 81 edges; its stored string uses a different labelling, and we verified an isomorphism to it. Both are among the twelve graphs that the House of Graphs returns for the keyword "Albertson-Berman", the keyword under which [3] says its graphs are deposited; that search also returns a 14-vertex graph with 36 edges and a 17-vertex graph with 45 edges (entries `57249` and `57250`), to which the two gadgets $G_{29}[A]$ and $G_{29}[B]$ of Proposition 2.1 below are isomorphic. Checked 2026-08-29.

TABLE 1. The adjacency of G_{29}. Degrees are $5^{16}6^{11}7^2$; there are 80 edges. G'_{29} is G_{29} with the edge 16 20 added.

$0 : 5, 6, 11, 12, 13$	$10 : 2, 4, 8, 17, 18$	$20 : 15, 22, 24, 26, 27, 28$
$1 : 2, 3, 4, 7, 9$	$11 : 0, 5, 13, 22, 24$	$21 : 15, 23, 25, 26, 27, 28$
$2 : 1, 3, 4, 8, 10$	$12 : 0, 6, 13, 23, 25$	$22 : 5, 11, 14, 20, 24, 26$
$3 : 1, 2, 7, 8, 19$	$13 : 0, 11, 12, 24, 25$	$23 : 6, 12, 14, 21, 25, 26$
$4 : 1, 2, 9, 10, 18$	$14 : 5, 6, 22, 23, 26$	$24 : 11, 13, 15, 20, 22, 25$
$5 : 0, 6, 11, 14, 22$	$15 : 20, 21, 24, 25, 27$	$25 : 12, 13, 15, 21, 23, 24$
$6 : 0, 5, 12, 14, 23$	$16 : 7, 9, 18, 19, 27, 28$	$26 : 14, 20, 21, 22, 23, 28$
$7 : 1, 3, 9, 16, 19$	$17 : 8, 10, 18, 19, 27, 28$	$27 : 15, 16, 17, 19, 20, 21, 28$
$8 : 2, 3, 10, 17, 19$	$18 : 4, 9, 10, 16, 17, 28$	$28 : 16, 17, 18, 20, 21, 26, 27$
$9 : 1, 4, 7, 16, 18$	$19 : 3, 7, 8, 16, 17, 27$	

In this labelling the identified edge is 27 28, and the two gadgets appear as

$$A = \{1, 2, 3, 4, 7, 8, 9, 10, 16, 17, 18, 19, 27, 28\},$$
$$B = \{0, 5, 6, 11, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25, 26, 27, 28\},$$

with $G_{29}[A] \cong Q$ and $G_{29}[B] \cong R$. The following is a finite check on Table 1, carried out by kernel evaluation.

Proposition 2.1. *$|A| = 14$, $|B| = 17$, $\big|E(G_{29}[A])\big| = 36$, $\big|E(G_{29}[B])\big| = 45$, $A \cup B = V$, $A \cap B = \{27, 28\}$, the pair $27, 28$ is an edge, and every edge of G_{29} has both ends in A or both ends in B.*

In particular $\{27, 28\}$ is a 2-cut of G_{29}: deleting it leaves the components $A \setminus B$ (twelve vertices) and $B \setminus A$ (fifteen vertices). This matches the vertex-connectivity 2 recorded for G_{29} in [3], where G'_{29} is recorded as having vertex-connectivity 3.

3. Two finite criteria for induced acyclicity

Acyclicity is a statement about all closed walks and is therefore not directly amenable to a finite check. The two lemmas of this section are textbook facts, but they are exactly the two directions a computation needs, one for each side of a value of a, and we record them because everything below is an instance of them. Both are proved for an arbitrary finite graph.

Lemma 3.1 (density)**.** *Let G be a graph and $T \subseteq V(G)$ nonempty with $G[T]$ a forest. Then*

$$\sigma_G(T) = \sum_{v \in T} d_T(v) \; < \; 2|T|,$$

that is, $G[T]$ has at most $|T| - 1$ edges.

Proof. $G[T]$ is acyclic and T is nonempty, so $G[T]$ extends to a spanning tree of the complete graph on T, which has exactly $|T| - 1$ edges; hence $|E(G[T])| \le |T| - 1$. The claim is the handshake identity. □

Lemma 3.2 (hereditary leaf)**.** *Let G be a graph and $S \subseteq V(G)$. If every nonempty $T \subseteq S$ contains a vertex v with $d_T(v) \le 1$, then $G[S]$ is a forest.*

Proof. Suppose $G[S]$ contained a cycle and let T be its vertex set. Every vertex of a cycle has exactly two neighbours on that cycle, so $d_T(v) \ge 2$ for all $v \in T$, contradicting the hypothesis at T. □

Lemma 3.2 certifies one explicit induced forest, by a search over the subsets of the candidate set. Lemma 3.1 is what an upper bound on a needs, in the following hereditary form: a set that is too dense cannot even sit *inside* an induced forest.

Corollary 3.3. *Let $C \subseteq V(G)$ be nonempty with $2|C| \le \sigma_G(C)$, i.e. every vertex of C has on average at least two neighbours inside C. Then no induced forest of G contains C.*

Proof. If $C \subseteq S$ and $G[S]$ is a forest, then so is $G[C]$, and Lemma 3.1 gives $\sigma_G(C) < 2|C|$. □

We call such a C a 2-*core certificate*; an induced cycle is the smallest example.

4. The forest number of the two graphs

The value in this section is due to Cames van Batenburg, Goedgebeur and Jooken [3]; what is presented here is a different certificate for it. Their argument observes that an induced forest F of G_{29} restricts to induced forests of the copies of Q and of R whose traces on the identified edge agree, so that with

$$p_J(X) = \max\bigl\{|F| : F \subseteq V(J),\ J[F] \text{ a forest},\ F \cap \{u,v\} = X\bigr\}$$

for a gadget J with distinguished edge uv, one gets $|F| \le \max_X \bigl(p_Q(X) + p_R(X) - |X|\bigr)$. The values, computed in [3], are

$$\bigl(p_Q(\emptyset), p_Q(\{x\}), p_Q(\{y\}), p_Q(\{x,y\})\bigr) = (6,7,7,7),$$
$$\bigl(p_R(\emptyset), p_R(\{b\}), p_R(\{c\}), p_R(\{b,c\})\bigr) = (8,8,8,9),$$

whence $a(G_{29}) \le \max(6+8,\ 7+8-1,\ 7+8-1,\ 7+9-2) = 14$.

We keep this decomposition and re-establish the two gadget bounds in the monotone form we need, using only Lemma 3.1 and Corollary 3.3. Write

$$\beta_A(S) = 6 + [\,27 \in S \text{ or } 28 \in S\,], \qquad \beta_B(S) = 8 + [\,27 \in S \text{ and } 28 \in S\,],$$

so that β_A collects p_Q and β_B collects p_R.

Table 2. The four 2-core certificates. Each is an induced cycle of G_{29}, so $\sigma_{G_{29}}(C) = 2|C|$ and every vertex has exactly two neighbours inside C. Together they span 22 vertices.

	vertices	shape	lives in
C_1	$\{3,4,7,8,9,10\}$	induced 6-cycle	A
C_2	$\{15,20,21,28\}$	induced 4-cycle	B
C_3	$\{6,11,12,13,14,22\}$	induced 6-cycle	B
C_4	$\{5,11,12,13,14,23\}$	induced 6-cycle	B

Proposition 4.1. *For every induced forest F of G_{29},*

$$|F \cap A| \le \beta_A(F \cap A) \qquad \textit{and} \qquad |F \cap B| \le \beta_B(F \cap B).$$

Proof sketch. This is where the computation sits. We check by exhaustive enumeration that every $S \subseteq A$ satisfies at least one of

$$|S| \le \beta_A(S), \qquad S \ne \emptyset \text{ and } 2|S| \le \sigma_{G_{29}}(S), \qquad C_1 \subseteq S, \tag{1}$$

and that every $S \subseteq B$ satisfies at least one of

$$|S| \le \beta_B(S), \qquad S \ne \emptyset \text{ and } 2|S| \le \sigma_{G_{29}}(S), \qquad C_2 \subseteq S, \qquad C_3 \subseteq S, \qquad C_4 \subseteq S. \tag{2}$$

These are two finite searches, over the $2^{14} = 16\,384$ subsets of A and the $2^{17} = 131\,072$ subsets of B; each condition is a Boolean function of the subset and no acyclicity test occurs in either. Given the searches, let F be an induced forest and put $S = F \cap A$. If the second alternative of (1) held, Lemma 3.1 applied to $S \subseteq F$ would give $\sigma_{G_{29}}(S) < 2|S|$, a contradiction; if the third held, then $C_1 \subseteq F$, contradicting Corollary 3.3 and the density of C_1 (Table 2). So the first alternative holds, which is the claim. The argument for B is identical with C_2, C_3, C_4. □

Remark 4.2. The density inequality alone does not suffice: our enumeration reports exactly five subsets that fall through it, one inside A and four inside B. The one inside A is the set $H = \{3,4,7,8,9,10,27,28\}$, for which $|H| = 8$, $\beta_A(H) = 7$, and $\sigma_{G_{29}}(H) = 14 < 16 = 2|H|$. The four inside B are

$$\{5,11,12,13,14,15,20,21,28\}, \quad \{6,11,12,13,14,15,20,21,28\},$$
$$\{6,11,12,13,14,15,21,22,28\}, \quad \{5,11,12,13,14,15,20,23,28\}.$$

Each of the five contains one of $C_1, \dots, C_4$, which is exactly why four certificates suffice. The formal development contains the case H as an explicit statement (Proposition 6.2 (v)); the list of the other four and the count "exactly five" are output of the search and are not part of the formal development, which only needs that every subset passes one of the alternatives in (1), (2).

Theorem 4.3. $a(G_{29}) = 14$. *That is, every induced forest of* G_{29} *has at most* 14 *vertices, and the set*

$$W = \{1,4,5,7,8,10,11,12,13,14,15,16,20,21\}$$

is an induced forest of order 14. *In particular* $2\,a(G_{29}) = 28 < 29 = |V(G_{29})|$.

Proof sketch. Upper bound. By Proposition 2.1, $F = (F \cap A) \cup (F \cap B)$ and $(F \cap A) \cap (F \cap B) = F \cap \{27,28\}$, so $|F| = |F \cap A| + |F \cap B| - |F \cap \{27,28\}|$. Feeding in Proposition 4.1 and distinguishing the four possibilities for $F \cap \{27,28\}$ gives $|F| \le \max(6+8,\ 7+8-1,\ 7+8-1,\ 7+9-2) = 14$.

Lower bound. $|W| = 14$, and $G_{29}[W]$ has the eleven edges $1\,4$, $1\,7$, $4\,10$, $5\,11$, $5\,14$, $7\,16$, $8\,10$, $11\,13$, $12\,13$, $15\,20$, $15\,21$; that it is a forest is certified by Lemma 3.2, through the finite search over the 2^{14} subsets $T \subseteq W$ verifying that each nonempty one has a vertex with $d_T(v) \le 1$. □

Theorem 4.4. $a(G'_{29}) = 14$, *where* $G'_{29} = G_{29} + 16\,20$. *In particular the planar triangulation* G'_{29} *on* 29 *vertices also violates Conjecture 1.1.*

Proof sketch. Since $G_{29} \subseteq G'_{29}$ on the same vertex set, an induced forest of G'_{29} is an induced forest of G_{29}, so the upper bound of Theorem 4.3 transfers verbatim. For the lower bound the same set W works: $G'_{29}[W]$ has the eleven edges of $G_{29}[W]$ together with $16\,20$, and the leaf search of Lemma 3.2 is rerun over the 2^{14} subsets of W in G'_{29}. □

Conjecture 1.1 concerns planar graphs, and planarity is not among the notions we formalise. Rather than commit to a definition, we state the refutation relative to an arbitrary predicate, so that the single unformalised input is a visible hypothesis; the certificate for that hypothesis is Section 5.

Theorem 4.5. *Let* Planar *be any predicate on finite simple graphs with* Planar(G_{29}). *Then it is not the case that every graph* G *with* Planar(G) *has an induced forest* S *with* $|V(G)| \le 2|S|$. *Moreover every induced forest* S *of* G_{29} *satisfies* $2|S| < 29$, *so* $c_{\mathcal{P}} \le 14/29 < 1/2$.

Proof. Immediate from Theorem 4.3: G_{29} satisfies the hypothesis and violates the conclusion, since $2 \cdot 14 = 28 < 29$. □

That Conjecture 1.1 is false is of course the result of [6, 7, 3] and we claim nothing for it here; Theorem 4.5 is stated only to make precise what the machine-checked development does and does not contain.

5. Planarity, isolated

A *rotation system* on a graph assigns to each vertex a cyclic ordering of its neighbours. Tracing faces — from the dart $u \to v$, step to $v \to w$ where w precedes u in the rotation at v — partitions the darts into orbits, the faces of the embedding determined by the rotation system, and Euler's formula $V - E + F = 2 - 2g$ reads off the genus of the supporting surface. Classically, a connected graph is planar exactly when some rotation system gives $V - E + F = 2$. We do not formalise that classical theorem; what we verify is its hypothesis, as an explicit combinatorial object.

Theorem 5.1. *There is a rotation system of G_{29} with 53 faces, so that $29-80+53=2$, and a rotation system of G'_{29} with 54 faces, so that $29-81+54=2$. In both cases the assignment is a genuine rotation system: at each vertex the list is a repetition-free enumeration of exactly that vertex's neighbours. Furthermore:*

(i) *the checker accepts K_4, with $4-6+4=2$;*
(ii) *transposing the first two entries of the rotation at vertex 0 of the system for G_{29} leaves a valid rotation system, with the same neighbour sets, whose face count is 51 and whose Euler characteristic is therefore 0, not 2;*
(iii) *$K_{3,3}$ has $(3!)^6 = 46\,656$ rotation systems, all valid, and* none *of them has Euler characteristic 2.*

Proof sketch. The rotation system for G_{29} is obtained from the straight-line drawing of Figure 3 of [3] by sorting each vertex's neighbours by the angle of the corresponding edge, after checking that no two vertex-disjoint edges of that drawing cross (2 793 pairs, all crossing-free). The system for G'_{29} inserts 20 into the rotation at 16 and 16 into the rotation at 20. Face tracing and the validity check are then evaluations on explicit arrays. The exhaustion in (iii) enumerates all 46 656 systems and traces the faces of each. □

Item (iii) is the guard that matters: the checker is not one that certifies everything, since it certifies no rotation system of a nonplanar graph. Item (ii) shows the certificate is sensitive to the cyclic data and not only to the neighbour sets — the perturbed system is an embedding of the same graph on the torus.

In the system for G'_{29} every face is a triangle, by counting: the 54 faces partition the $2 \cdot 81 = 162$ darts, and no face orbit is shorter than three, since a shorter one would force a loop or a vertex of degree at most 1 and G'_{29} is simple of minimum degree 5; so all 54 orbits have length exactly three. For G_{29} the same count leaves one unit of slack, and our face tracer — an implementation independent of the Lean one, see Section 10 — reports the shapes explicitly: 52 triangles and a single quadrilateral face, 16 27 20 28. Its two diagonals are 27 28, already an edge, and 16 20; so $G'_{29} = G_{29} + 16\,20$ is the only triangulation obtainable from this embedding by adding one edge, which is consistent with the uniqueness statement of [3].

6. Controls

A refutation is worth nothing if the machinery that produced it rejects graphs satisfying the conjecture, or accepts sets that are not forests. The icosahedron I is the natural positive control, and the values below are again those stated in [3], in the sentence motivating their gadgets; we re-derive them by the criteria of Section 3. We label I with 0 the north pole, $1, \ldots, 5$ the upper pentagon, $6, \ldots, 10$ the lower pentagon and 11 the south pole.

Theorem 6.1. *For the icosahedron I we have $a(I) = 6 = |V(I)|/2$ and $\alpha(I) = 3 = |V(I)|/4$. In particular I satisfies the Albertson–Berman bound with equality, and the criteria of Section 3* accept *it.*

Proof sketch. I is 5-regular on 12 vertices, hence has 30 edges. The upper bound $a(I) \le 6$ is a search over all $2^{12} = 4096$ subsets, verifying that each S satisfies $|S| \le 6$ or is dense enough for Lemma 3.1 to exclude it; no 2-core certificate is needed here. The set $\{0, 1, 4, 6, 7, 9\}$ is an induced path 7 6 1 0 4 9 of order 6, certified by Lemma 3.2. For α, the set $\{3, 5, 6\}$ is independent, and a search over all 2^{12} subsets finds no independent set of order 4. □

Proposition 6.2. *The following hold.*

(i) *There is no induced forest of G_{29} on 15 vertices (the value is not too large).*
(ii) *The bound $a(G_{29}) \le 13$ is false (the value is not too small), and W is an induced forest of order exactly 14, so the extremal class is nonempty.*
(iii) *The set $\{0, 1, \ldots, 13\}$ also has 14 vertices and is* not *an induced forest: it contains C_1.*

(iv) *Each of* C_1, C_2, C_3, C_4 *satisfies* $\sigma_{G_{29}}(C_i) = 2|C_i|$ *with every vertex having exactly two neighbours inside* C_i*; they are induced cycles.*

(v) *The certificates are load-bearing:* $H = \{3, 4, 7, 8, 9, 10, 27, 28\}$ *satisfies* $H \subseteq A$, $|H| = 8$, $\beta_A(H) = 7$, $\sigma_{G_{29}}(H) = 14 < 16 = 2|H|$ *and* $C_1 \subseteq H$*. Thus* H *passes the density test and violates the gadget bound; the search* (1) *would fail without* C_1*.*

(vi) *Both gadget bounds are attained simultaneously:* $|W \cap A| = 6 = \beta_A(W \cap A)$ *and* $|W \cap B| = 8 = \beta_B(W \cap B)$*, with* $27, 28 \notin W$*.*

Item (v) is the sharp one. It exhibits a subset of the Q-gadget on which the hereditary density inequality alone is strictly weaker than the gadget bound; without C_1 the enumeration (1) does not go through, so the four certificates of Table 2 are not decoration.

7. Bounded-degree induced forests

We now prove Theorem 1.3. The whole content is the following observation about the witness W of Theorem 4.3.

Lemma 7.1. W *is a* linear *induced forest of* G_{29} *and of* G'_{29}*: every* $v \in W$ *has* $d_W(v) \leq 2$*.*

Proof. Direct evaluation of $d_W(v)$ for the fourteen vertices of W in each of the two graphs. □

Concretely, reading the components off the eleven edges of $G_{29}[W]$ listed in the proof of Theorem 4.3, $G_{29}[W]$ is the disjoint union of the three paths

$$16\,7\,1\,4\,10\,8, \qquad 14\,5\,11\,13\,12, \qquad 20\,15\,21,$$

and $G'_{29}[W]$, which adds the edge $16\,20$, is the disjoint union of the two paths

$$8\,10\,4\,1\,7\,16\,20\,15\,21, \qquad 14\,5\,11\,13\,12.$$

Only the degree bound of Lemma 7.1 is used below.

Theorem 7.2. *For every integer* $d \geq 2$*,*

$$f_d(G_{29}) = f_d(G'_{29}) = 14.$$

Equivalently, $f_d(G_{29}) = a(G_{29})$ *and* $f_d(G'_{29}) = a(G'_{29})$ *for all* $d \geq 2$*: on these two graphs the bounded-degree restriction costs nothing.*

Proof. For the lower bound, W has order 14 and, by Lemma 7.1, maximum degree at most $2 \leq d$ inside itself, in G_{29} and in G'_{29} alike; so it is admissible for f_d in both. For the upper bound, an induced forest of maximum degree at most d is in particular an induced forest, so $f_d \leq a = 14$ by Theorems 4.3 and 4.4. □

Corollary 7.3. *Let* $d \geq 2$ *be an integer and* $n = 29$*.*

(i) *If* $d \geq 7$ *then* $f_d(G_{29}) \leq \frac{2dn}{4d+1}$ *and* $f_d(G'_{29}) \leq \frac{2dn}{4d+1}$*: both graphs are counterexamples to Conjecture 1.2.*

(ii) *If* $2 \leq d \leq 6$ *then* $f_d(G_{29}) > \frac{2dn}{4d+1}$ *and* $f_d(G'_{29}) > \frac{2dn}{4d+1}$*: both graphs satisfy Conjecture 1.2.*

Proof. By Theorem 7.2 both quantities equal 14, and over the integers $2d \cdot 29 < (4d+1) \cdot 14$ reads $58d < 56d + 14$, i.e. $d < 7$. □

Proposition 7.4. *The threshold is exact, not an artefact of rounding: at* $d = 6$ *one has* $2 \cdot 6 \cdot 29 = 348 < 350 = (4 \cdot 6 + 1) \cdot 14$*, while at* $d = 7$ *the two sides are equal,*

$$2 \cdot 7 \cdot 29 \;=\; 406 \;=\; (4 \cdot 7 + 1) \cdot 14.$$

Equivalently $29 = 4 \cdot 7 + 1$ *and* $14 = 2 \cdot 7$*, so* $a(G_{29})/29 = 2d/(4d+1)$ *holds with equality at* $d = 7$*, and what* G_{29} *violates at* $d = 7$ *is precisely the strictness of the inequality in Conjecture 1.2.*

Remark 7.5. Table 3 compares the ranges of d that the published counterexamples to Conjecture 1.1 certify for Conjecture 1.2 through the inequality $f_d \leq a$. For every graph but G_{29} and G'_{29} the entry is only a guaranteed range: the values f_d of those graphs are not published, so a smaller d is not excluded for them. For G_{29} and G'_{29}, Corollary 7.3 makes the range exact in both directions. The smallest order in the table from which the full range $d \geq 7$ is reached drops from 52 to 29.

TABLE 3. Counterexamples to Conjecture 1.1 and the range of d for which $a(G) \leq 2d\,|V(G)|/(4d+1)$, hence for which they refute Conjecture 1.2.

graph	order	a	refutes Conjecture 1.2 for
Jung [6]	31	15	$d \geq 8$
Makarov [7]	39	19	$d \geq 10$
Makarov [7]†	76	37	$d \geq 10$
G_{41} of [3]	41	20	$d \geq 10$
25/52 family of [3]	52	25	$d \geq 7$
G_{29}, G'_{29} (here)	29	14	$d \geq 7$, and only these

† [7] indicates this variant without a full proof.

Remark 7.6. The following counts were produced by a separate implementation in C and are not part of the formal development. The graph G_{29} has 36 444 induced forests of order 14, of which 8 904 have maximum degree 2, 26 220 have maximum degree 3 and 1 320 have maximum degree 4; G'_{29} has 30 644, split 7 262 / 23 382 between maximum degrees 2 and 3. So linear maximum induced forests are not rare in G_{29} — roughly a quarter of them — which is the quantitative reason the degree restriction in Theorem 7.2 is free. Nothing in Theorem 7.2 or Corollary 7.3 depends on these numbers; the single set W suffices.

8. Vertex-criticality

The following is not new: it is implied by the minimality theorem of [3], since every one-vertex deletion of G_{29} is a planar graph on 28 vertices and therefore satisfies Conjecture 1.1. We record it because the implication passes through the 10 CPU-year computation of [3], whereas for this one graph the same local conclusion takes three certificates.

Theorem 8.1. *For every vertex v of G_{29} there is an induced forest of G_{29} of order 14 avoiding v. Consequently $a(G_{29}-v) = 14 = |V(G_{29}-v)|/2$ for every v: every one-vertex deletion of G_{29} satisfies the Albertson–Berman bound with equality, so deleting any vertex destroys the counterexample. Three sets suffice for all 29 vertices, namely W together with*

$$W_2 = \{0, 2, 3, 6, 7, 9, 15, 17, 18, 22, 23, 24, 26, 27\},$$
$$W_3 = \{0, 1, 3, 8, 9, 10, 11, 12, 14, 16, 20, 21, 23, 24\}.$$

Proof sketch. W_2 and W_3 are induced forests of order 14, each certified by Lemma 3.2 through a search over its own 2^{14} subsets. A finite check confirms that every $v \in V(G_{29})$ lies outside at least one of W, W_2, W_3. The upper bound $a(G_{29} - v) \leq a(G_{29}) = 14$ is Theorem 4.3. The three sets were found by a greedy cover over the maximum induced forests enumerated in Remark 7.6; the pair W, W_2 would not do on its own, since 7 and 15 lie in both, and it is W_3 that avoids them. □

9. Open questions

Question 9.1. Conjecture 1.2 for $2 \leq d \leq 6$. Corollary 7.3 says the minimum-order counterexample does not settle any of these cases. The natural next probe is Jung's 31-vertex triangulation [6], for which $a = 15$ and the arithmetic of Remark 7.5 gives only $d \geq 8$: computing f_d for it would decide whether the extra two vertices buy a smaller d. By the leaf-peeling dynamic program of Section 10, that is a sweep over 2^{31} subsets, about two gigabytes and a few minutes. Makarov's 39-vertex graph [7] is 2^{39} and out of reach by the same route; a feedback-vertex-set solver would be needed.

Question 9.2. The same analysis for G_{41}, the unique 4-connected 5-edge-connected counterexample of minimum order [3], with $a(G_{41}) = 20$ and $41 = 4 \cdot 10 + 1$, $20 = 2 \cdot 10$ — so G_{41} sits exactly on the Conjecture 1.2 threshold at $d = 10$, as G_{29} does at $d = 7$. Whether $f_d(G_{41}) = 20$ for all $d \geq 2$ is the same question one level up. A flat sweep over 2^{41} subsets is out of reach, but G_{41} is covered by two copies of $Q - xy$ and one copy of $R - bc$, pairwise meeting in the two endpoints of the distinguished edge, with three further edges outside all three (Figure 4 of [3]), so the gadget reduction of Section 4 should apply with a larger interface.

Question 9.3. A human-checkable certificate for $a(G_{29}) = 14$. The authors of [3] remark that their gadget values can also be verified by a short case analysis, using the descriptions of Q and R as icosahedron modifications. The four certificates of Table 2 shrink the exceptional set to 22 vertices, but the density inequality is still swept over $2^{14} + 2^{17}$ subsets. Replacing that sweep by an argument would make the whole refutation readable end to end.

Question 9.4. Making the counts of Remark 7.6 machine-checked. What is needed is the splitting lemma: F induces a forest of G_{29} if and only if $F \cap A$ and $F \cap B$ both do. The forward direction is immediate; the converse holds because a cycle meeting both private sides would cross $\{27, 28\}$ twice and, with the edge 27 28, force a second 27–28 path inside one gadget.

Two further things sit outside the formal development, for different reasons. The exhaustive rotation-system control for K_5 — all $(4!)^5 = 7\,962\,624$ systems, none spherical, at most 5 faces — runs in 2.6 seconds in C, but the corresponding streaming evaluation did not finish inside our 30-minute budget; the $K_{3,3}$ control of Theorem 5.1(iii) is a complete nonplanarity guard on its own. And the minimality half of [3] — that no counterexample of order at most 28 exists — is six orders of magnitude beyond what we can run, and is used here only as a citation.

10. Verification

Every theorem, proposition, lemma and corollary in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; the development contains no `sorry`, and the repository reference is to be added at submission. The material that lies outside the development is stated as such where it occurs: the edge and path lists displayed in the proof of Theorem 4.3 and after Lemma 7.1, which are read off Table 1; the counts of Remarks 4.2 and 7.6; the face shapes of G_{29} reported after Theorem 5.1; and the exhaustive K_5 control of Section 9. The two criteria of Section 3, Lemmas 3.1 and 3.2, are proved for an arbitrary finite graph and depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`; so do Corollary 3.3, Theorem 4.5 given Theorem 4.3, and Theorem 6.1's consequence that the icosahedron satisfies the Albertson–Berman bound. Proposition 2.1, the density of the four certificates of Table 2, Proposition 6.2, the degree bound of Lemma 7.1, Proposition 7.4 and the covering check of Theorem 8.1 are kernel evaluations on the explicit data of Table 1. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches and nothing else: the 2^{14} and 2^{17} sweeps (1) and (2) behind Proposition 4.1; the 2^{14} leaf searches certifying W in G_{29} and in G'_{29} and W_2, W_3 in G_{29}; the 2^{12} sweeps behind $a(I) \leq 6$ and $\alpha(I) \leq 3$; and the face tracing of Theorem 5.1, including the enumeration of all 46 656 rotation systems of $K_{3,3}$. The whole development compiles in about 92 seconds.

Every numerical value quoted was first produced by an independent implementation in C or Python, and the formal proof was written against it afterwards. In particular $a(G_{29}) = 14$ and $a(G'_{29}) = 14$ were each reproduced by a leaf-peeling dynamic program over all 2^{29} subsets (46 seconds and 526 megabytes per graph), the gadget values $p_Q = (6, 7, 7, 7)$ and $p_R = (8, 8, 8, 9)$ of [3] by sweeps over 2^{14} and 2^{17} subsets, and $a(I) = 6$, $\alpha(I) = 3$ by a sweep over 2^{12} subsets; the face counts and the quadrilateral face 16 27 20 28 were reproduced by an independent face-tracing implementation. The graph data itself was extracted from the figure source of [3] and cross-checked: the degree sequence, the edge count 80, the gadget split of Proposition 2.1 and the isomorphisms of $G_{29}[A]$ with Q and $G_{29}[B]$ with R respecting the distinguished edge were all confirmed before any formal proof was attempted.

All references to [3], in particular the figure numbers, are to version 1 of that preprint, submitted on 24 August 2026. Its arXiv listing showed no later version, no journal reference and no DOI other than the arXiv one when we last checked it, on 29 August 2026.

References

[1] M. O. Albertson and D. M. Berman, *A conjecture on planar graphs*, in: J. A. Bondy and U. S. R. Murty (eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, p. 357.

[2] O. V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979) 211–236.

[3] W. Cames van Batenburg, J. Goedgebeur and J. Jooken, *Counterexamples to the Albertson–Berman conjecture: minimum order, connectivity and an improved ratio bound*, arXiv:2608.23260v1 (2026).

[4] G. G. Chappell and M. J. Pelsmajer, *Maximum induced forests in graphs of bounded treewidth*, Electron. J. Combin. **20** (2013), no. 4, #P8.

[5] K. Coolsaet, S. D'hondt and J. Goedgebeur, *House of Graphs 2.0: A database of interesting graphs and more*, Discrete Appl. Math. **325** (2023) 97–107; `https://houseofgraphs.org`.

[6] H. Jung, *A* 15/31 *Counterexample Family to the Albertson–Berman Conjecture*, arXiv:2608.17350v1 (2026).

[7] M. Makarov, *A counterexample to the Albertson–Berman conjecture about induced forests in planar graphs*, arXiv:2608.13964v1 (2026).

[8] B. Mohar, *Induced forests in planar graphs*, Problem of the Month, August 2002. Available at `https://www.sfu.ca/~mohar/Problems/P0208InducedForestPlanar.html`.

[9] D. B. West, *Induced forests in planar graphs*, in: Problems in Graph Theory and Combinatorics. Available at `https://dwest.web.illinois.edu/openp/planforest.html`.