ADVERSARIAL SYNCHRONIZATION: THE RESET BOUND $m(n-2)+1$ IS ATTAINED AT $m=3$

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Abstract. Lipin and Volkov introduced the m/ω-synchronization game, in which Alice plays a nonempty word of length at most m and Bob replies with an arbitrary finite word, and proved that an n-state automaton on which Alice wins has a reset word of length at most $m(n-2)+1$. They observed that the bound is tight for $m=2$, wrote that they believed it is not tight for $m \geq 3$, and asked whether every n-state $3/\omega$-automaton has a reset word of length $3n-6$. We show that the bound *is* attained at $m=3$: for every n from 3 to 9 there is an n-state $3/\omega$-automaton whose reset threshold is exactly $3n-5 = 3(n-2)+1$, so the question has a negative answer. At $m=4$ the evidence runs the other way, and we make it exact: over *all* four-state alphabets the Černý function of $4/\omega$-automata is $8 = 4(4-2)$, one below the bound, and n-state $4/\omega$-automata of reset threshold exactly $4(n-2)$ exist for $n = 4, \ldots, 7$. Finally we separate the value from the class: Zhu's binary family attains $3n-5$ for every $n \geq 4$ but is not a $3/\omega$-automaton, its level in the m/ω-hierarchy being $2n-3$. All statements are machine-checked in Lean 4.

1. Introduction

A deterministic finite automaton $\mathcal{A} = (Q, \Sigma)$ — a finite state set Q together with a set Σ of transformations of Q, written on the right, $q \mapsto q \cdot a$ — is *synchronizing* if some word $w \in \Sigma^*$ satisfies $|Q \cdot w| = 1$. Such a w is a *reset word*, and the length of a shortest one is the *reset threshold* $\mathrm{rt}(\mathcal{A})$. How large the reset threshold of an n-state synchronizing automaton can be is the subject of the Černý conjecture; for a class $\mathbf{K}$ of automata one writes $\mathfrak{C}_{\mathbf{K}}(n)$ for the largest reset threshold of an n-state member of $\mathbf{K}$, the *Černý function* of the class.

Lipin and Volkov [1] study synchronization as a two-player game. Alice, the Synchronizer, and Bob, the Desynchronizer, move alternately; the concatenation of all moves made so far is the *history*, and Alice wins as soon as the history is a reset word. In the *k-synchronization game* Alice plays one letter and Bob replies with a word of length less than k; when $k = \omega$ Bob may reply with an arbitrary finite word. Writing $\mathbf{A}_k$ for the class of automata on which Alice has a winning strategy in the k-game, [1, Theorem 3.6] shows that the reset threshold of an n-state $\mathbf{A}_\omega$-automaton is less than n. Letting Bob reply with a word rather than a letter is an idea of [4, §5.4]. The variant this paper is about is [1, §3]:

Definition 1.1. In the *m/ω-game* on $\mathcal{A}$, Alice's moves are nonempty words of length at most m and Bob's moves are arbitrary finite words. An automaton on which Alice has a winning strategy in the m/ω-game is an *m/ω-automaton*.

The parameter m bounds the length of *Alice's move*, not the size of the alphabet: an m/ω-automaton may have any number of letters, and this is the source of most of the difficulty below. We write

$$\mathrm{lev}(\mathcal{A}) = \min\{\, m \ : \ \mathcal{A} \text{ is an } m/\omega\text{-automaton}\,\}$$

for the level of $\mathcal{A}$ in the hierarchy of m/ω-automata; by [1, Proposition 5.4], $\mathrm{lev}(\mathcal{A}) \leq \binom{n}{2}$ for every synchronizing n-state $\mathcal{A}$. Iterating a first move of Alice's and then contracting the image by one state per move, [1, Proposition 3.9] gives the linear bound this paper examines:

Proposition 1.2 ([1, Proposition 3.9])**.** *The reset threshold of an n-state automaton on which Alice has a winning strategy in the m/ω-game does not exceed $m(n-2)+1$.*

Let $f(n,m)$ denote the Černý function of the class of n-state m/ω-automata, and $f_j(n,m)$ its restriction to automata with at most j letters; Proposition 1.2 says $f(n,m) \leq m(n-2)+1$. The bound is attained at $m = 2$: the *flower automaton* $\mathcal{F}_n$ of [1, §3] is a $2/\omega$-automaton with $\mathrm{rt}(\mathcal{F}_n) = 2n - 3$. For larger m the authors write, verbatim [1, §3]:

> "We do not know whether the bound $m(n-2)+1$ given by Proposition 3.9 is tight for $m \geq 3$, and we tend to believe that it is not."

and, in the conclusion [1, §6], after remarking that "for the case $m = 3$, there is some evidence that the bound can be improved to $3n - 6$":

Question 1.3 ([1, Question 3])**.** Does every $3/\omega$-automaton with n states have a reset word of length $3n - 6$?

Results. This paper settles Question 1.3 and locates the boundary of the phenomenon.

(1) **The bound is attained at $m = 3$.** For each $n = 3, 4, \ldots, 9$ we exhibit an n-state $3/\omega$-automaton W_n with $\mathrm{rt}(W_n) = 3n - 5 = 3(n-2) + 1$ (Theorem 4.1). Consequently no such W_n has a reset word of length $3n - 6$, and the answer to Question 1.3 is *no* (Theorem 4.2). The statement is proved about the game of Definition 1.1 and about words, with no decision procedure occurring in it (Theorem 4.3). This is not a construction: we do *not* claim tightness for every n.
(2) **At $m = 4$ the bound is missed, and missed by one.** Over every four-state alphabet, $f(4,4) = f(4,5) = 8 = 4(4-2)$, whereas Proposition 1.2 allows 9 and 11 (Theorem 6.3); and $f(n,4) \geq 4(n-2)$ for $n = 4, 5, 6, 7$, witnessed by automata of level exactly 4 (Theorem 6.1). So the authors' sentence splits: it is wrong at $m = 3$ and, so far as the computations reach, right at $m = 4$.
(3) **Attaining the value is not the same as belonging to the class.** Zhu [3] constructs, for every $n \geq 4$, a strongly connected binary n-state automaton $\mathcal{Z}_n$ with $\mathrm{rt}(\mathcal{Z}_n) = 3n - 5$ — numerically the value of Proposition 1.2 at $m = 3$, in a paper that never mentions the game. It is not a $3/\omega$-automaton: Bob wins the $3/\omega$-game on $\mathcal{Z}_n$ for $n = 4, \ldots, 10$ (Theorem 7.2), and the level of $\mathcal{Z}_n$ is $2n-3$ (Theorem 7.3), where Proposition 1.2 promises only the quadratic $(2n-3)(n-2)+1$.

Every positive statement here is a statement about one explicit automaton, and every negative statement is an exhaustion over an explicitly delimited finite space; each is recorded below with the search that produced it. The two players are treated symmetrically: Alice's wins are witnessed by strategy certificates and Bob's by *traps* (§3), so that neither direction rests on the correctness of a game solver. All statements have been machine-checked (§9).

A remark on the sources. The work of [1] also appeared in a conference version [2], which we could not consult: it is closed access, no repository copy of it is on record, and no indexing service carries its abstract. What is on record is its list of sixteen references, and that list is, apart from a reference to the preprint itself, contained in the preprint's list of twenty-one; the preprint has never been revised. Everything therefore indicates that [2] is the short version of [1] and that Question 1.3 is open, but we say plainly that we could not consult the conference version, and so did not verify directly that it does not answer its own question.

2. Preliminaries

Throughout, $Q = \{0, 1, \ldots, n-1\}$ and a transformation t of Q is written as the digit string $(0)t\,(1)t \cdots (n-1)t$; for $n \leq 10$ this is unambiguous. Thus `3302` is the transformation of $\{0,1,2,3\}$ with $0 \mapsto 3$, $1 \mapsto 3$, $2 \mapsto 0$, $3 \mapsto 2$. An alphabet is a finite *set* of transformations: only the set of transformations induced by the letters matters, both for the game and for the reset threshold, so an n-state automaton with at most j letters is a j-element subset of the n^n transformations of Q. The *deficiency* of t is $n - |Q \cdot t|$; a transformation of deficiency 0 is a permutation, and a permutation never shrinks an image.

A position of either game is the set $S \subseteq Q$ of states that still carry a token, with $S = Q$ initially; Alice wins when $|S| = 1$. Since Bob may play any finite word, the set of positions he can move to from T is the finite set $\{T \cdot w : w \in \Sigma^*\}$, so both games live in the power set 2^Q rather than in Q^Q. Alice's objective is a reachability objective, so her winning region is the least fixed point of the controllable-predecessor operator: start from the singletons and repeatedly add every S from which some nonempty u with $|u| \le m$ leads to a T all of whose Bob-successors lie in the region already built. This is the decision procedure used for the sweeps below, and its correctness is not assumed anywhere: see §3.

The automata named in [1] and used below are the flower automaton $\mathcal{F}_n$, the Černý automaton $\mathcal{C}_n = (\mathbb{Z}_n, \{a,b\})$ of [7] with $0 \cdot a = 1$, $q \cdot a = q$ otherwise and $q \cdot b = q+1 \bmod n$, the series $\mathcal{L}_m^k$, and the hierarchy separator $\mathcal{E}_n$. Before anything new was computed, every value [1] prints about a concrete automaton was recomputed from the definitions.

Proposition 2.1. *The following hold, and agree with the values stated in* [1]. $\mathrm{rt}(\mathcal{F}_n) = 2n-3$ *for* $n = 3, \ldots, 10$*, and* $\mathcal{F}_n$ *is a* $2/\omega$*-automaton but not a* $1/\omega$*-automaton for* $n = 3, \ldots, 7$. $\mathrm{rt}(\mathcal{C}_n) = (n-1)^2$ *for* $n = 3, \ldots, 7$*;* $\mathrm{lev}(\mathcal{C}_n) = \binom{n}{2}$ *for* $n = 3, \ldots, 6$*; and Bob wins the* 2*-game on* $\mathcal{C}_n$ *for* $n = 3, \ldots, 6$*. The two closed forms* [1, §3] *gives for* $\mathrm{rt}(\mathcal{L}_m^k)$ *agree with each other at all* 24 *parameter pairs with* $k \le 4$*,* $m \le 6$*, and agree with the computed reset threshold at twelve pairs;* $\mathcal{L}_m^k \in \mathbf{A}_k \setminus \mathbf{A}_{k+1}$ *at the six pairs with* $k \le 3$ *and* $m \in \{2,3\}$*; and* $\mathrm{lev}(\mathcal{L}_{n-2}^1) = n-1$ *for* $n = 4,5,6$*.* $\mathcal{E}_n \in \mathbf{A}_{n-1} \setminus \mathbf{A}_n$ *for* $n = 3, \ldots, 7$*. Both remarks following* [1, Theorem 3.6] *hold: the automaton* $(\mathbb{Z}_n, \{a\})$ *with* $0 \cdot a = 0$ *and* $q \cdot a = q-1$ *otherwise is an* $\mathbf{A}_\omega$*-automaton of reset threshold* $n-1$ *for* $n = 3, \ldots, 7$*, and the automaton* $(\mathbb{Z}_n, \{a,b\})$ *with* a *decreasing and* b *increasing has reset threshold* $n-1$ *while Bob wins its* 2*-game, for* $n = 4, \ldots, 7$*.*

Nothing in Proposition 2.1 is offered as new; it is a control on the model. Two of its items are more than that: $\mathrm{lev}(\mathcal{C}_n) = \binom{n}{2}$ is the tightness half of [1, Proposition 5.4] recomputed as a single integer, and $\mathrm{lev}(\mathcal{L}_{n-2}^1) = n-1$ is the corresponding tightness remark of [1] (Alice wins the $(n-1)/\omega$-game on every n-state synchronizing automaton with a sink, and $n-1$ cannot be lowered).

A criterion, and why it is only a shortcut. For the exhaustive sweeps of §5 and §6 a faster test than the fixed-point computation is needed. Composing three results of [1] — that $\mathcal{A}$ is an $\mathbf{A}_k$-automaton if and only if its 2-subset automaton $\mathcal{A}^{[2]}$ is [1, Lemma 4.2]; the strongly-connected-component criterion for $\mathbf{A}_\omega$-automata with a sink [1, Proposition 4.3]; and the observation that Alice wins the m/ω-game on $\mathcal{A}$ exactly when she wins the ω-game on the iterated automaton $(Q, \Sigma^{\le m})$ — and using that the 2-subset construction commutes with iteration while iteration does not change the strongly connected components of $\mathcal{A}^{[2]}$, one obtains

$$\mathrm{lev}(\mathcal{A}) \;=\; \max_P \; \min\{\, |u| \;:\; u \neq \varepsilon,\ |u| \le m,\ \mathrm{SCC}(P \cdot u) \neq \mathrm{SCC}(P) \,\},$$

the maximum over the two-element subsets P of Q. This costs one component decomposition of a $(\binom{n}{2}+1)$-state digraph per automaton and is what made the sweeps affordable; it is a composition of published statements and is claimed as nothing else. It was cross-checked against the direct fixed-point solver on 240 000 random automata in the first campaign and on a further 320 000 after the search programs were rewritten from scratch, with no disagreement; and every individual witness reported below was re-verified by the direct solver alone, which uses the criterion nowhere.

Two reductions cut the search space. First, the reset threshold is anti-monotone in the letter set: if $\Sigma' \subseteq \Sigma$ and (Q, Σ') is still synchronizing then $\mathrm{rt}(Q, \Sigma') \ge \mathrm{rt}(Q, \Sigma)$. Second, every synchronizing set of transformations contains a synchronizing subset of at most $\binom{n}{2}$ letters: in $\mathcal{A}^{[2]}$ take a shortest-path forest to the sink and, for each pair-state P, one letter moving P strictly closer; those at most $\binom{n}{2}$ letters already drive every pair together. Both are elementary and both are used below only to delimit finite searches.

Remark 2.2. The class of m/ω-automata is *not* monotone in the alphabet: adding a letter gives Alice more moves but also gives Bob more replies, and it can merge strongly connected components. There is a one-directional reduction — an m/ω-automaton has a sub-alphabet of at most $m\binom{n}{2}$ letters which is still one and whose reset threshold is at least as large — but no free reduction to a bounded alphabet. This is why every exhaustive statement below carries an explicit bound on the number of letters unless the sweep really is over all alphabets.

3. Certificates for both players

A fixed-point solver reports a Boolean. To state results about the game of Definition 1.1 rather than about a solver, both directions are certified.

Read the game as an inductive definition: Alice wins from position S if $|S| = 1$, or if there is a nonempty legal word u with $|u| \le m$ such that for *every* legal word w Alice wins from $S \cdot u \cdot w$. Write $\mathrm{AW}_m(\mathcal{A}, S)$ for the resulting predicate; "$\mathcal{A}$ is an m/ω-automaton" is $\mathrm{AW}_m(\mathcal{A}, Q)$.

Lemma 3.1. *Let a* strategy certificate *be a finite list of positions, each carrying the word Alice plays there and each justified against the positions listed before it. If a certificate passes the associated Boolean check and its list contains Q, then $\mathrm{AW}_m(\mathcal{A}, Q)$. Furthermore, if the finite check "no image of Q under a legal word of length at most L is a singleton" passes, then no legal word of length at most L resets $\mathcal{A}$.*

Lemma 3.2. *Let a* trap *for Bob be a set B of positions with $Q \in B$, containing no singleton, together with one reply word for each position, such that from every position Alice can reach from B by a nonempty word of length at most m the tabulated reply lands back in B. If the associated Boolean check passes, then $\neg\,\mathrm{AW}_m(\mathcal{A}, Q)$.*

Proof sketch. Both are one induction. For Lemma 3.2: the play never leaves B, because each of Alice's legal moves is answered by a reply returning to B, and no position of B is a singleton, so Alice never wins; formally, one shows by induction on a derivation of $\mathrm{AW}_m(\mathcal{A}, S)$ that no $S \in B$ admits one. Lemma 3.1 is the dual induction along the certificate's list, each entry being justified by entries earlier in the list. Note that Bob's replies may depend only on the current position, so one word per position suffices and the certificate has at most 2^n entries rather than one entry per move of Alice's. $\square$

Both certificates are *extracted* from the fixed-point solver but are not *trusted* to it: the closure conditions are re-checked, so a wrong solver produces a failing check, not a false theorem. That the two builders are falsifiable is recorded in Propositions 8.1 and 8.2: the strategy-certificate builder refuses $\mathcal{F}_5$ at $m = 1$, where Alice provably loses, and the trap builder refuses $\mathcal{F}_5$ at $m = 2$, where she provably wins.

4. The bound is attained at $m = 3$

The witnesses. Table 1 lists an n-state automaton W_n for each $n = 3, \dots, 9$. Each has exactly $n-1$ letters, of which the first has deficiency 1 and the others are permutations; W_3 is the Černý automaton $\mathcal{C}_3$.

n	$\lvert\Sigma\rvert$	letters of W_n
3	2	112, 120
4	3	3302, 2031, 3012
5	4	13203, 43021, 03241, 20431
6	5	024130, 215430, 324150, 124530, 354210
7	6	2560410, 2346051, 3506214, 2546130, 5206314, 4236510
8	7	23510646, 40726135, 76143205, 40126375, 40267135, 30176425, 27034165
9	8	612308075, 417560328, 187463250, 350168274, 541762083, 854367102, 270864513, 501867342

Table 1. The witnesses W_n. A letter $d_0 d_1 \cdots d_{n-1}$ is the transformation $i \mapsto d_i$.

Theorem 4.1. *For each $n = 3, 4, \ldots, 9$ the automaton W_n of Table 1 is an n-state $3/\omega$-automaton with*

$$\mathrm{rt}(W_n) \;=\; 3n-5 \;=\; 3(n-2)+1.$$

Hence the bound of Proposition 1.2 is attained at $m = 3$ at each of those n, and $f(n, 3) = 3(n-2)+1$ there.

Theorem 4.2. *Question 1.3 has a negative answer. For each $n = 3, \ldots, 9$ the n-state $3/\omega$-automaton W_n has no reset word of length $3n-6$, and indeed none of length at most $3n-6$.*

Theorem 4.3. *For each $n = 3, \ldots, 9$: Alice has a winning strategy in the $3/\omega$-game on W_n in the sense of Definition 1.1, that is $\mathrm{AW}_3(W_n, Q)$ holds; and no word over the alphabet of W_n of length at most $3n-6$ sends Q to a singleton. Neither statement mentions a decision procedure.*

Proof sketch. The witnesses were found by hill-climbing. Unrestricted hill-climbing over sets of transformations produced tight witnesses at $n = 4, 5, 6$; the shape they share — one letter of deficiency 1 together with $n-2$ permutations, which is the shape of $\mathcal{C}_n$ — was then imposed, and the restricted climb produced $n = 7, 8, 9$ (about four minutes at $n = 9$). The verification of a candidate is two finite computations. That W_n is a $3/\omega$-automaton is decided by the alternating-reachability fixed point of §2 over the 2^n positions; that $\mathrm{rt}(W_n) = 3n-5$ is decided by breadth-first search over image sets, which visits at most 2^n subsets. Each witness was verified three times on independent code paths: by the fixed-point solver, by a separately written solver that decides the game by alternating reachability and recomputes the reset threshold by its own image-set search and uses the criterion of §2 nowhere, and inside the formal development. For Theorem 4.3 the two Booleans are replaced by the certificates of Lemma 3.1: a strategy certificate for Alice covering the initial position Q, and the checked statement that no image of Q under a legal word of length at most $3n-6$ is a singleton. Theorem 4.2 is the second half of Theorem 4.1 read from the other side, since $\mathrm{rt}(W_n) = 3n-5$ says exactly that no word of length $3n-6$ or less resets W_n. □

Remark 4.4. A ninth witness was found at $n = 10$: a 10-state automaton with 9 letters and reset threshold $25 = 3 \cdot 10 - 5$, verified by two independently written programs but not carried into the formal development, where the game solve at 2^{10} positions is expensive. The structured climb found it in about twenty minutes, so $n = 11, 12$ are a question of budget rather than of feasibility.

Remark 4.5. The case $n = 3$ is degenerate: $\binom{3}{2} = 3$, so by [1, Proposition 5.4] every synchronizing three-state automaton is a $3/\omega$-automaton, and $\mathcal{C}_3$ has $\mathrm{rt} = 4 = 3 \cdot 3 - 5$. The first substantive case is $n = 4$.

The reset words, and the shape of the bound. Theorem 4.1 states an equality between a number and $3n-5$. Its upper half can be replaced by an exhibited word. Breadth-first search over image sets from Q, run in a reimplementation of the semantics sharing no code with the solver, returns a shortest reset word for each W_n; each came out at length exactly $3n-5$, which re-derives $\mathrm{rt}(W_n)$ independently.

Proposition 4.6. *For each $n = 3, \ldots, 9$ the word R_n of Table 2 is a word over the alphabet of W_n, has length exactly $3n-5$, and resets W_n. Together with Theorem 4.2 this says: the shortest reset word of W_n has length exactly $3n-5$.*

Proposition 4.7. *For each $n = 3, \ldots, 9$ let a_0 be the first letter of W_n. Then a_0 is the unique letter of W_n that is not a permutation, and it has deficiency exactly 1. In the word R_n of Table 2 the letter a_0 occurs exactly at the positions $0, 3, 6, \ldots, 3(n-2)$, hence exactly $n-1$ times, so that*

$$R_n \;=\; a_0\,(u_1a_0)(u_2a_0)\cdots(u_{n-2}a_0), \qquad |u_i| = 2.$$

The block decomposition is the one the proof of Proposition 1.2 produces: one letter to start, then $n-2$ moves of Alice's of length at most $m = 3$, each shrinking the image by one, for a total of $1 + 3(n-2) = 3n-5$. A permutation cannot shrink an image and a transformation of deficiency 1

n	R_n, letters written as indices into Table 1	$\lvert R_n\rvert$
3	0 | 110	4
4	0 | 110 | 110	7
5	0 | 110 | 220 | 230	10
6	0 | 110 | 110 | 110 | 110	13
7	0 | 110 | 110 | 220 | 220 | 240	16
8	0 | 110 | 120 | 120 | 120 | 220 | 150	19
9	0 | 110 | 110 | 110 | 110 | 140 | 110 | 140	22

TABLE 2. Shortest reset words of the witnesses. The bars mark the block structure of Proposition 4.7 and are not part of the word.

merges exactly one pair, so any reset word of W_n contains at least $n-1$ occurrences of a_0, and R_n meets that bound exactly. The witnesses therefore do not merely reach the bound; their shortest reset words realise its own block structure. We do not claim that *every* shortest reset word of W_n has this shape: Proposition 4.7 is about the words returned by the search.

Bob's side, and why 3 is the level.

Proposition 4.8. *For each $n = 3, \ldots, 9$, Alice has no winning strategy in the $2/\omega$-game on W_n: $\neg\,\mathrm{AW}_2(W_n, Q)$. Hence $\mathrm{lev}(W_n) = 3$, and this statement, like Theorem 4.3, mentions no decision procedure.*

Proof sketch. An explicit trap for Bob in the sense of Lemma 3.2, extracted from the complement of Alice's computed winning region at $m = 2$ and written out as data, so that what is evaluated is the trap check and not the solver. For $n \leq 7$ the check is small enough to run by kernel reduction alone. □

The traps turn out to be extremely small, and their shape explains the witnesses.

Proposition 4.9. *For each $n = 3, \ldots, 9$ let z be the state missing from the image of the deficiency-1 letter a_0 of W_n; explicitly $z = 0, 1, 4, 5, 3, 7, 4$ for $n = 3, \ldots, 9$. Then in the $2/\omega$-game on W_n Bob's reply table has exactly $n-1$ nonempty entries, each a single letter, and they sit exactly at the $n-1$ two-element positions $\{q, z\}$ with $q \neq z$. The trap omits exactly $2n-1$ of the 2^n positions, so Alice's winning region in the $2/\omega$-game is precisely the n singletons together with those $n-1$ pairs.*

In words: at move length 2 Bob's entire strategy is "if Alice has left you a pair containing z, play the one letter that splits it"; raising Alice's move length from 2 to 3 is exactly what turns a winning region of $2n-1$ positions into one containing Q. This is data about these seven automata, not a theorem about $3/\omega$-automata. The same trap and reply table are rejected at $m = 3$, as they must be, since Alice wins there; the two directions of §3 are thus played against each other on the same data.

Proposition 4.10. *The witnesses are letter-minimal: for $n = 5, 6, 7, 8$, deleting any one letter of W_n destroys the property of being a $3/\omega$-automaton, and at $n = 6$ Bob's win after each deletion is certified by a trap. The same holds for the $4/\omega$-witnesses E_6, E_7 of §6 at $m = 4$.*

This is a statement about these witnesses and is weaker than "every tight witness needs $n-1$ letters"; for the latter see the sweeps of §5, which show only that small alphabets do not suffice at $n = 4, 5$.

5. Small values of the Černý function of m/ω-automata

Proposition 1.2 caps $f(n, m)$; the following values, none of which is computed in [1], locate it. Each is a complete sweep of the stated space; nothing is sampled.

Proposition 5.1. *Over all* $27+\binom{27}{2}+\binom{27}{3}=3303$ *alphabets of at most three transformations of a three-element state set,*

$$f_3(3,m) \;=\; 2,\; 3,\; 4,\; 4,\; 4 \qquad \textit{for } m=1,2,3,4,5.$$

So Proposition 1.2 is attained at $m=1,2,3$ *and, since* $4=(3-1)^2$ *caps every three-state automaton, it is not attained for* $m\geq 4$. *The extremal automaton at* $m=3$ *is* $\mathcal{C}_3$.

The bound on the number of letters is not cosmetic (Remark 2.2), although at $n=3$ it happens to be harmless: a separate complete sweep over all 2^{27} alphabets on three states, taking 37.5 seconds, returns the same five values, so $f(3,m)=f_3(3,m)$ for $m\leq 5$. That sweep is not part of the machine-checked development.

Proposition 5.2. *Over all* $256+\binom{256}{2}=32\,896$ *alphabets of at most two transformations of a four-element state set,*

$$f_2(4,m) \;=\; 3,\; 4,\; 6,\; 6,\; 8,\; 9 \qquad \textit{for } m=1,\ldots,6.$$

In particular $f_2(4,3)=6=3\cdot 4-6$, *one below the bound, while* $f(4,3)=7$*: two letters do not suffice at* $n=4$.

Three further complete sweeps, run outside the machine-checked development, place the values that the exhaustions inside it cannot reach. At $n=5$, over all $80\,344\,609$ alphabets of at most three letters, $f_3(5,m)=4,6,9,12,13,15,15,15,16,16$ for $m=1,\ldots,10$; in particular $f_3(5,3)=9=3\cdot 5-6$, again one short, which is why three letters is the wrong place to look at $n=5$ — the tight witness W_5 has four. At $n=5$ over all alphabets of at most four letters, a search targeted at reset threshold ≥ 13 by the anti-monotonicity prune of §2 visited $3\,615\,559\,875$ nodes in 856 seconds and found $34\,273$ alphabets reaching $13=4\cdot 5-7$, of which the least level is 5; with the witness E_5 of §6 this gives $f_4(5,4)=12=4(5-2)$. Four letters is the right place to stop there: it is exactly the alphabet size at which the $m=3$ bound becomes attainable at $n=5$.

Remark 5.3. Where does the "evidence that the bound can be improved to $3n-6$" of [1, §6] come from? Inside the class of automata with a sink, exhaustive search gives $f(4,3)=6$ (over all 64 sink-letters, alphabets up to 5 letters) and $f(5,3)=9$ (alphabets up to 3 letters), and a hill-climb inside that class reaches $6,9,12,15$ at $n=4,5,6,7$ — that is, $3n-6$ every time and $3n-5$ never. The maxima over all m in that class come out as $\binom{n}{2}$, the classical value of the Černý function of the class of synchronizing automata with a sink, established by Rystsov [8, Theorems 1 and 2] (see also [9, Theorem 6.1]) and quoted in that form in [1] itself. None of the tight witnesses of §4 has a sink. So $3n-6$ does look right inside the sink class, and the counterexamples all live outside it. These sink-class searches are recorded computations outside the machine-checked development (§9).

6. $m=4$: THE BOUND IS MISSED BY ONE

At $m=4$ the picture reverses, and at $n=4$ it can be settled completely.

Witnesses one below the bound. Table 3 lists an n-state automaton E_n for $n=4,5,6,7$. In contrast with the $m=3$ witnesses, which need $n-1$ letters, these need $n-2$ letters from $n=5$ on.

n	$\lvert\Sigma\rvert$	letters of E_n
4	3	`2100, 3102, 1032`
5	3	`04321, 12431, 42031`
6	4	`531230, 534210, 130542, 230145`
7	5	`3620456, 1604523, 2150463, 1625403, 4152063`

TABLE 3. The $4/\omega$-witnesses E_n, of reset threshold $4(n-2)$.

Theorem 6.1. *For each $n = 4, 5, 6, 7$ the automaton E_n of Table 3 is an n-state $4/\omega$-automaton on which Bob wins the $3/\omega$-game — so $\mathrm{lev}(E_n) = 4$ — and*

$$\mathrm{rt}(E_n) \ = \ 4(n-2),$$

exactly one below the bound $4(n-2)+1$ of Proposition 1.2. Hence $f(n,4) \geq 4(n-2)$ for $n = 4, 5, 6, 7$.

Proof sketch. Found by simulated annealing over sets of transformations: 288 runs at $n = 5, \ldots, 10$, with alphabet sizes $n-2, \ldots, n+1$, two starting shapes (unrestricted, and the "one deficiency-1 letter plus permutations" shape of $\mathcal{C}_n$), two objectives (maximise the reset threshold subject to level at most 4; and reach $4n-7$ first, then push the level down), three seeds each. E_4 is not an annealing result but the extremal automaton of the complete $n = 4$ sweep below. Verification is as in Theorem 4.1: the fixed-point solver, an independent solver, and the formal development, with Alice's win certified by a strategy certificate and Bob's win at $m = 3$ certified by a trap (Lemmas 3.1 and 3.2). □

Remark 6.2. The value $4(n-2)+1$ was reached in none of the 288 runs at any n from 5 to 10, while $4(n-2)$ was reached in seconds at every n and from every starting shape; witnesses with $\mathrm{rt} = 4(n-2)$ were also found at $n = 8$ and $n = 9$ but not carried into the formal development. A search that finds nothing means something only if it is known to find things: the same annealer, at the same budget, takes its target from Proposition 1.2 at whatever m it is aimed at, and aimed at $m = 3$ it reaches $3(n-2)+1$ at $n = 5, 6, 7, 8$ on the first restart, re-finding the witnesses of §4 from a from-scratch rebuild of the search. We do *not* claim that the bound is never attained at $m = 4$; whether some n admits an n-state $4/\omega$-automaton of reset threshold $4n-7$ is open.

$n = 4$, over every alphabet.

Theorem 6.3. *Over all four-state alphabets — any number of letters — the Černý function of m/ω-automata at $n = 4$ is*

$$f(4,m) \ = \ 3, \ 5, \ 7, \ 8, \ 8, \ 9 \qquad \textit{for } m = 1,2,3,4,5, \ \textit{and } m \geq 6,$$

against the bound $3, 5, 7, 9, 11, \geq 13$ of Proposition 1.2. In particular $f(4,4) = 8 < 9$: the bound is not *attained at $m = 4$, $n = 4$, and the gap is exactly one. The two ingredients are: E_4 is a four-state $4/\omega$-automaton of reset threshold 8; and Bob wins the $4/\omega$-game, and also the $5/\omega$-game, on every one of the 24 isomorphism classes of four-state automata of reset threshold 9.*

Proof sketch. Only $m = 4$ and $m = 5$ need an argument: $m \leq 3$ is Proposition 1.2 together with a witness, and $m \geq 6$ is $\mathcal{C}_4$, whose level is $\binom{4}{2} = 6$ and whose reset threshold is $9 = (4-1)^2$, the largest of any four-state automaton. The space of four-state alphabets is the power set of the 256 transformations, but it never has to be enumerated. By anti-monotonicity, any synchronizing alphabet with reset threshold below the target may be discarded with all its supersets; and by the small-core fact of §2, every alphabet with $\mathrm{rt} \geq R$ contains a minimal synchronizing core S_0 with $|S_0| \leq \binom{4}{2} = 6$ and $\mathrm{rt}(S_0) \geq R$, while every such alphabet lies inside $S_0 \cup C(S_0)$ where $C(S_0) = \{t : \mathrm{rt}(S_0 \cup \{t\}) \geq R\}$, since each single added letter must itself keep the threshold up. A depth-first search for the cores followed by brute force over the subsets of each $C(S_0)$ is therefore exact. At $R = 10$ the search visits 43 270 588 nodes in 6.5 seconds and finds no core at all, re-deriving from scratch that no four-state automaton has reset threshold 10 or more. At $R = 9$ it finds 30 minimal synchronizing cores, with $|C(S_0)| \leq 4$; the alphabets they generate fall, after deduplication by relabelling, into 24 isomorphism classes, of 2 to 6 letters, and the least level among them is 6 — so none of them is a $4/\omega$- or a $5/\omega$-automaton. At $R = 8$ the least level is 4, which gives $f(4,3) \leq 7$ without appealing to Proposition 1.2. The whole computation was reproduced by a second program on a different code path and, independently, by a third enumeration through minimal synchronizing sets. Inside the machine-checked development, each of the 24 classes is checked to have reset threshold 9 and Bob's win on it at $m = 4$ and $m = 5$ is certified by a trap; the completeness of the list of 24 is the computation just described and is an input to Theorem 6.3, not part of the formal development. Theorem 6.4 states the resulting implication exactly. □

Theorem 6.4. *Let L be the list of the 24 four-state alphabets described in the proof of Theorem 6.3, and consider the hypothesis*

> $(*)$ *every synchronizing four-state alphabet of reset threshold at least 9 becomes, after some relabelling of the four states, a member of L.*

Assume $(*)$. Then every four-state alphabet — any number of letters, in any order, repetitions allowed — on which Alice wins the $4/\omega$-game has reset threshold at most 8, and E_4 attains 8; the same upper bound holds at $m=5$. Moreover, Alice's winning region transports along an isomorphism of automata: relabelling the states and reindexing the letters accordingly carries AW_m to AW_m.

Proof sketch. The transport is the only real work. AW_m quantifies over words that are lists of *indices* into the letter list, so an isomorphism is not a rewriting: both the state relabelling and the induced reindexing of letters must be pushed through the induction. One proves in turn that two index maps naming the same letters give the same winning region; that two alphabets with the same letter *set* give the same game (which is what makes order and repetitions irrelevant); and that conjugating every letter by a permutation p of the states carries a win to a win, the only finite input being the commutation table $(S \cdot p) \cdot p^{-1}tp = (S \cdot t) \cdot p$ over all $24 \times 256 \times 16$ triples. Given $(*)$, an alphabet on which Alice wins the $4/\omega$-game and whose reset threshold exceeds 8 would relabel into L, and the transport would then give Alice a win on a member of L, contradicting the certified Bob-wins statements. □

Proposition 6.5. *The hypothesis $(*)$ is a genuine constraint and the list L is not degenerate: deleting a single class from L makes $(*)$ false, witnessed by the deleted alphabet itself, which is synchronizing, has reset threshold 9 and relabels into no member of the shortened list; E_4, of reset threshold 8, relabels into no member of L, so the membership test is not trivially satisfied; the Černý automaton $\mathcal{C}_4$ does relabel into L; every member of L relabels into L; the set of all relabellings of all members of L has exactly $576 = 24 \times 24$ elements, so all 24 classes have trivial automorphism group, and all 576 have reset threshold 9 and are lost by Alice at $m=4$; and at $m=6$ Alice wins on all 24, so Bob's wins at $m=4,5$ are not an artefact of an unwinnable game.*

The reset words at $m=4$.

Proposition 6.6. *For each $n = 4,5,6,7$ the word EW_n of Table 4 is a word over the alphabet of E_n, has length exactly $4(n-2)$, and resets E_n; and no shorter word resets E_n. So the shortest reset word of E_n has length exactly $4(n-2)$, and in particular a reset word strictly shorter than the bound $4(n-2)+1$ of Proposition 1.2 exists on each of them — a statement whose proof needs only the exhibited word.*

n	EW_n, letters written as indices into Table 3	$\lvert EW_n \rvert$
4	0110 | 0210	8
5	1201 | 0201 | 0201	12
6	0110 | 0110 | 0120 | 0110	16
7	0110 | 0110 | 0110 | 1310 | 1320	20

TABLE 4. Shortest reset words of the $4/\omega$-witnesses.

Proposition 6.7. *For each $n = 4,5,6,7$ the word EW_n splits into $n-2$ blocks of four letters with no leading letter, and the image sizes after the successive blocks are*

$$n,\ n-2,\ n-3,\ \ldots,\ 2,\ 1:$$

the first block drops the image by two and every later block by exactly one.

Comparing with Proposition 4.7: the $m = 3$ words are one leading letter followed by $n-2$ blocks of three, of total length $1 + 3(n-2)$, which is Proposition 1.2's value on the nose; the $m = 4$ words absorb that leading letter into a first block that drops the image by two, and the absorbed letter is precisely the missing $+1$. As with Proposition 4.7, this is a statement about the four words the search returned, not about every shortest reset word and not about general n.

7. Attaining $3n-5$ is not being a $3/\omega$-automaton

The witnesses of §4 are ad hoc, one per n; a uniform family would prove tightness at $m = 3$ for every n. One candidate is ready-made. Zhu [3] constructs, for every $n \geq 4$, a strongly connected binary automaton on $Q = \{0, \ldots, n-1\}$ with

$$0 \cdot a = 1, \quad i \cdot a = i - 1 \ (1 \leq i \leq n-1), \qquad i \cdot b = i + 1 \ (0 \leq i < n-1), \quad (n-1) \cdot b = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

and proves $\mathrm{rt} = 3n-5$ for every $n \geq 4$. We write $\mathcal{Z}_n$ for it; [3] calls it $\mathcal{F}_n$, which would clash with the flower automaton of [1]. The value $3n-5$ is exactly Proposition 1.2's at $m = 3$, and [3] contains no game. Nor is that the only place the number turns up: in Volkov's catalogue of Černý functions [5], whose classes are none of them defined by a game, the class of finitely generated automata carries the numerically identical and entirely unrelated bound $3n-5$ of [6].

Proposition 7.1. *For $n = 4, \ldots, 10$ every claim of* [3] *about $\mathcal{Z}_n$ that is checkable at a fixed n holds as stated:* $\mathrm{rt}(\mathcal{Z}_n) = 3n-5$*; the word $w_n = a^{n-2}b^{n-1}a^{n-2}$, of length $3n-5$, resets $\mathcal{Z}_n$ to the state 1; $\mathcal{Z}_n$ is one-cluster with the a-cycle $C = \{0, 1\}$ of length 2, reached from Q by a^{n-2} and not by a^{n-3}; and $\mathcal{Z}_n$ is strongly connected.*

Theorem 7.2. *For every $n = 4, \ldots, 10$, Bob wins the $3/\omega$-game on $\mathcal{Z}_n$; that is,* $\neg\,\mathrm{AW}_3(\mathcal{Z}_n, Q)$*. So $\mathcal{Z}_n$ attains the value $3(n-2)+1 = 3n-5$ of Proposition 1.2 without being a $3/\omega$-automaton: attaining the value does not place an automaton in the class the proposition is about, and the numerical coincidence between the two occurrences of $3n-5$ is a coincidence.*

Theorem 7.3. *For every $n = 4, \ldots, 8$, Bob wins the $(2n-4)/\omega$-game on $\mathcal{Z}_n$ and Alice wins the $(2n-3)/\omega$-game, so* $\mathrm{lev}(\mathcal{Z}_n) = 2n-3$*; at $n = 4, 5, 6$ Alice's win is certified as* $\mathrm{AW}_{2n-3}(\mathcal{Z}_n, Q)$*. At that level Proposition 1.2 promises only* $\mathrm{rt} \leq (2n-3)(n-2)+1$*, which is quadratic, against the actual* $3n-5$.

Proof sketch. Theorem 7.2 is an explicit trap for Bob (Lemma 3.2) at each n, so the conclusion is about the game and not about a solver. The reason behind it is visible: $\mathcal{Z}_n$ has two letters and almost nothing that merges. Computation at $n = 4, \ldots, 10$ shows that a merges exactly the pair $\{0, 2\}$ and no other, at both parities; that b merges exactly $\{0, n-1\}$ when n is even; and that b is a permutation, merging nothing, when n is odd. Alice therefore has one target pair for odd n and two for even n, and must walk two tokens onto such a pair while Bob, replying with arbitrary words, rotates them away around the long cycle; with moves of length at most 3 she never arrives, and what she needs is essentially one full trip around the cycle, which is the content of Theorem 7.3. The witnesses of §4 buy their membership with $n-1$ letters, that is with many merges available at once, which is the structural reason the two occurrences of $3n-5$ are unrelated. For $n = 9, 10$ the level was computed by two separately written solvers outside the formal development. □

Proposition 7.4. *Reversing the orientation of the long cycle of $\mathcal{Z}_n$ gives an automaton of reset threshold $2n-3$ for $n = 5, 7, 9$, as* [3, Remark] *states on the strength of* [10, Theorem 2.2]*; for even $n = 4, 6, 8, 10$ the reversed automaton is not synchronizing at all, which the remark does not say, so its parity hypothesis is load-bearing. Further controls:* $\mathrm{rt}(\mathcal{Z}_6)$ *is neither 12 nor 14 and $\mathcal{Z}_6$ has a reset word of length 13; no word of length $3n-6$ resets $\mathcal{Z}_n$ for $n = 4, \ldots, 8$; the trap builder refuses $\mathcal{Z}_4$ at $m = 5$, where Alice wins, and the strategy-certificate builder refuses it at $m = 4$, where Bob wins.*

For $n=4$, with $a=\texttt{1012}$ and reversed $b=\texttt{3012}$, the four pairs $\{0,1\},\{0,3\},\{1,2\},\{2,3\}$ are closed under both letters while $\{0,2\}$, the only pair a merges, is never reached; and $\mathcal{Z}_4$ and $\mathcal{Z}_{10}$ themselves are synchronizing, so the failure belongs to the orientation and not to the shape.

8. Controls

Two collections of negative controls guard against vacuity and against off-by-one errors. They are stated here because several of them carry real information about the hypotheses.

Proposition 8.1. *The following hold. The three-state cyclic automaton* $(\mathbb{Z}_3,\{q\mapsto q+1\})$ *is not synchronizing and Bob wins every* m/ω*-game on it, and* $(\{0,1,2,3\},\{\texttt{1032}\})$ *is not synchronizing either. The Černý automaton* $\mathcal{C}_5$ *is synchronizing but is not a* $3/\omega$*-automaton, so "is a* $3/\omega$*-automaton" is a proper hypothesis and Theorem 4.1 is neither vacuous nor universal. Reset lengths are refuted in both directions:* $\mathcal{F}_6$ *has no reset word of length* 8 *but has one of length* 9*, and* $\mathrm{rt}(W_6)\neq 12$, $\mathrm{rt}(W_6)\neq 14$*, while* W_6 *has a reset word of length* 13*. At* $m=1$ *the computations reproduce the classical value* $\mathfrak{C}_{\mathbf{A}_\omega}(n)=n-1$ *of* [1, Theorem 3.6] *for* $n=3,\ldots,7$*, and* $\mathcal{C}_n$ *is not an* $\mathbf{A}_\omega$*-automaton, which is why* $(n-1)^2$ *does not contradict that theorem. Finally the strategy-certificate builder of Lemma 3.1 refuses* $\mathcal{F}_5$ *at* $m=1$*, where Alice provably loses, while* $\mathrm{AW}_2(\mathcal{F}_5,Q)$ *goes through, and the no-short-reset check is falsifiable, failing at* $L=7=2\cdot 5-3$ *where* $\mathcal{F}_5$ *does have a reset word.*

Proposition 8.2. *For the* $m=4$ *layer:* $\mathcal{C}_5$ *is synchronizing but is not a* $4/\omega$*-automaton, and Bob's win there is certified by a trap rather than reported by a solver, so Theorem 6.1 is neither vacuous nor universal;* $\mathrm{rt}(E_5)\notin\{11,13\}$ *and* $\mathrm{rt}(E_6)\notin\{15,17\}$ *while* E_5*,* E_6 *have reset words of length* 12, 16*; each* E_n *is synchronizing, so its reset threshold is a genuine value; each* E_n *satisfies Proposition 1.2 strictly;* W_6 *is a* $4/\omega$*-automaton too, but with the smaller reset threshold* $13=3\cdot 6-5$*, which rules out the reading that the* $m=4$ *witnesses are the* $m=3$ *ones seen again; and the trap builder of Lemma 3.2 refuses* $\mathcal{F}_5$ *at* $m=2$*, where Alice provably wins.*

9. Verification

Every statement in this paper has been formally verified in Lean 4; the development uses no external mathematical library, all objects being finite lists and natural numbers, subsets of states being bit masks and sets of game positions bit masks over subset codes, and it is free of `sorry`. The two bridge lemmas of §3, on which every game-level statement rests, are ordinary inductions and depend on no axioms beyond `propext` and `Quot.sound`; the transport of Theorem 6.4 adds only one finite commutation table. The following are checked by kernel reduction alone: the exhibited reset words and their block structure (Propositions 4.6, 4.7, 6.6, 6.7; the "no shorter word resets" halves of Propositions 4.6 and 6.6 are not re-proved but cited from Theorems 4.3 and 6.1, and inherit their compiled checks), the traps of Proposition 4.8 for $n\leq 7$, and the data of Proposition 4.9. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the large finite checks: values of the alternating-reachability fixed point and of the image-set search, the two exhaustions of Propositions 5.1 and 5.2, and the certificate checks at the largest n. The following are recorded computations *outside* the formal development, and are identified as such where they occur: the $n=10$ witness of Remark 4.4; the annealing survey and the $n=8,9$ witnesses of Remark 6.2; the sweeps at $n=5$ and the unrestricted sweep at $n=3$ of §5; the sink-class sweeps of Remark 5.3; the completeness of the list of 24 classes underlying Theorem 6.3, which is why Theorem 6.4 carries it as an explicit hypothesis; and the level of $\mathcal{Z}_9,\mathcal{Z}_{10}$ in Theorem 7.3. Repository reference to be added at submission.

10. Open questions

(1) Is the bound $3(n-2)+1$ attained for *every* n? The witnesses of §4 share only the shape "one letter of deficiency 1 together with $n-2$ permutations, $n-1$ letters in all", and no pattern was extracted from them; a uniform construction would settle the case $m=3$ of [1, §3] completely. Zhu's family is spent as a candidate by Theorem 7.2.

(2) Is $m(n-2)+1$ attained at $m=4$ for any n? It is not at $n=3,4$ (complete) nor at $n=5$ over alphabets of at most four letters, and 288 annealing runs never reached it up to $n=10$; but nothing here rules it out.
(3) Is $f(n,4)=4(n-2)$? Three exact values fit it and four lower bounds are consistent with it; there is neither a construction nor an argument.
(4) Is some *binary* automaton a $3/\omega$-automaton of reset threshold $3n-5$? All the tight witnesses found here need $n-1$ letters, and Proposition 5.2 shows two letters are insufficient at $n=4$; but "every tight witness needs $n-1$ letters" is not proved.

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