ACTIVE PAIRS OF A MIN–MAX PROBLEM ON ZERO-SUM PLANAR CONFIGURATIONS: THE HOUSE PENTAGON, THE 3-4 RECTANGLE, AN OPTIMUM THAT IS NOT ACTIVE-MAXIMAL, AND HOW MANY PAIRS A CONFIGURATION CAN ACTIVATE

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Abstract. For $z_1, \dots, z_n \in \mathbb{C}$ summing to zero put $\mathrm{pv}(z_i, z_j) = |z_i| + |z_j| + |z_i + z_j|$ and $\delta(z) = \max_{i<j} \mathrm{pv}(z_i, z_j) / \sum_k |z_k|$; the quantity $\pi^{\max}_{n,2} = \inf_z \delta(z)$, the infimum taken over the z with every $z_k \neq 0$, measures where the eigenvalue set of the locally positive semidefinite matrices fails to be convex, and in the polygon dictionary of Acevedo, Blekherman, Debus, Lee and Riener it asks how small the largest two-edge triangle perimeter of a convex n-gon can be relative to its perimeter. A pair (i, j) is *active* when it attains the maximum. We certify, in exact integer arithmetic, the two configurations those authors conjecture to be optimal: the *house* pentagon has perimeter 112 and largest two-edge triangle perimeter 80, so $\delta = 5/7$ exactly and $\pi^{\max}_{5,2} \le 5/7$, with 7 of its 10 pairs active; the 3-4 *rectangle* at $n = 2m$, $m \ge 2$, has perimeter $6m + 2$ and maximum 12, so $\delta = 4/(n + \frac{2}{3})$ exactly and $\pi^{\max}_{n,2} \le 4/(n + \frac{2}{3})$, with $m^2 + m - 2$ of its $2m^2 - m$ pairs active — the last uniformly in m, not one n at a time. We prove the comparison $4/(x + \frac{2}{3}) < (4/x)\cos^2(\pi/2x)$ for every real $x \ge 5$ that those authors assert in one clause, so that the regular n-gon is never optimal for even $n \ge 6$, and we pin their "approximately 1.3%" at the pentagon to the exact algebraic number $(5\sqrt{5} - 11)/14 \in (0.0128, 0.0129)$. A uniform bound $\delta(z) \ge 2/n$, improved here to $\delta(z) \ge 1/c_n$ with $c_n = (n-1) - \sqrt{(n-1)(n-2)/2}$, makes these upper bounds non-empty; the best lower bound in the literature is those authors' own $4/(n+1)$, so the live window at $n = 5$ is $2/3 \le \pi^{\max}_{5,2} \le 5/7$. Our first new result answers the second half of their closing open question at $n = 4$, in the negative: the square, which they prove to be the unique optimal configuration there, activates 4 pairs, while the non-trivial zero-sum configuration $(1, -1, 0, 0)$ activates 5. The phenomenon is confined to $n = 4$.

We then answer the first half of that question for $n \le 4$ and begin a structure theory valid at every n. At $n = 3$ every zero-sum triple activates all three pairs. At $n = 4$ the maximum over their non-trivial class is exactly 5, and a configuration attains it only if it is a permutation of $(u, -u, 0, 0)$ with $u \neq 0$; over the configurations with no vanishing entry the maximum is exactly 4, attained by the square and by the 3-4 rectangle. Over the configurations having an entry with $2|z_k| = \max_{i<j} \mathrm{pv}$ — which we show are exactly the collinear ones, with $\delta = 1$ — the maximum is exactly $2n - 3$ at every $n \ge 3$. Outside that boundary case every entry acquires a plane vector representative in which activity is the single equation $\langle W_i, W_j \rangle^2 = 1$; two structure lemmas follow, that an entry with $4|z_k| < \max_{i<j} \mathrm{pv}$ is active with at most two entries, and that an entry lying strictly inside the cone spanned by an active non-parallel pair satisfies $4|z_k| < \max_{i<j} \mathrm{pv}$. Finally we exhibit, at every odd $n = 2g+1$, an explicit configuration with $g^2 + g$ active pairs, so that $2n - 3$ is not the odd-n maximum from $n = 7$ on; its δ gives $\pi^{\max}_{n,2} < (4/n)\cos^2(\pi/2n)$ for every odd $n \ge 7$, with $\pi^{\max}_{7,2} \le (15 + \sqrt{57})/42$, where those authors report only that their experiments suggest the regular polygon to be suboptimal and exhibit no configuration. Every theorem below is machine-checked in Lean 4; the development uses no floating-point arithmetic and no compiled evaluation.

1. Introduction

The objects. A *configuration* of size n is a tuple $z = (z_1, \dots, z_n)$ of complex numbers with $\sum_k z_k = 0$. Write

$$\mathrm{pv}(a, b) = |a| + |b| + |a + b|, \qquad \mathrm{mass}(z) = \sum_{k=1}^{n} |z_k|, \qquad \mathrm{pmax}(z) = \max_{1 \le i < j \le n} \mathrm{pv}(z_i, z_j), \tag{1}$$

and $\delta(z) = \mathrm{pmax}(z) / \mathrm{mass}(z)$. Both pmax and mass are homogeneous of degree one and invariant under a global rotation, so δ is a scale- and rotation-invariant functional on configurations, and the min–max problem

$$\pi^{\max}_{n,2} = \inf\{\delta(z) :\ z \text{ a configuration of size } n \text{ with every } z_k \neq 0\} \tag{2}$$

is a problem about shapes. A pair (i, j) with $i < j$ is called *active* at z when $\mathrm{pv}(z_i, z_j) = \mathrm{pmax}(z)$, and we write $\mathrm{act}(z)$ for the number of active pairs.

The quantity (2) is due to Acevedo, Blekherman, Debus, Lee and Riener [1], who arrive at it from a spectral question. Let $\lambda(S^{n,2})$ be the set of eigenvalue vectors of the real symmetric $n \times n$ matrices all of whose 2×2 principal submatrices are positive semidefinite. That set is not convex, and [1] shows that $\pi^{\max}_{n,2}$ measures where it fails to be (their Problem 3.1, Proposition 3.3 and Remark 3.4), reaching (2) through a "complex squaring" reduction that also appears, for a different functional, in Nesterenko [2].

What makes (2) a geometry problem is the dictionary of [1, §4]. Sorted by argument, the numbers $z_1, \dots, z_n$ of a configuration are the edge vectors of a closed convex polygonal path $A_1A_2 \cdots A_n$; then $\mathrm{mass}(z)$ is the perimeter of that polygon, and for two edge vectors z_i, z_j the number $\mathrm{pv}(z_i, z_j)$ is the perimeter of the triangle with sides z_i, z_j and $-(z_i + z_j)$ — for consecutive indices, the triangle $A_kA_{k+1}A_{k+2}$ cut off by the diagonal A_kA_{k+2}. So $\pi^{\max}_{n,2}$ asks: *over all convex n-gons, how small can the largest two-edge triangle perimeter be, relative to the perimeter of the polygon?*

What [1] conjectures, and what it leaves open. Section 4 of [1] is a list of conjectures and open problems, and three of its items are the starting point here. For $n = 5$:

> "*The house*, corresponding to a convex pentagon $ABCDE$ where $ABCE$ is a rectangle and $|AB| = 30$, $|BC| = |EA| = 16$, $|CD| = |DE| = 25$ (see Figure 3a), is an optimal polygon with $\pi^{\max}_{5,2} = 5/7$. We have verified that the house is a strict local minimum up to rotation and scaling." [1], Conjecture 4.3

For even n, the 3-4 *rectangle*: take $(n-2)/2$ of the z_k equal to 3, as many equal to -3, one equal to $4i$ and one equal to $-4i$; then $\pi^{\max}_{n,2} = 4/(n + \frac{2}{3})$ for even $n \geq 6$ [1, Conjecture 4.4]. Both conjectures have two halves. The *value* half — that the stated configuration has the stated δ — is an identity about one explicit object. The *optimality* half — that no configuration does better — is a statement about all of $\mathbb{C}^n$. Since $\pi^{\max}_{n,2}$ is an infimum, the value half alone already gives an unconditional *upper* bound on $\pi^{\max}_{n,2}$; the reverse inequality is the conjecture.

The third item is [1, Remark 4.5], which closes its §4. Having observed that both conjecturally optimal configurations activate roughly half of all pairs — the house 7 of $\binom{5}{2} = 10$, the 3-4 rectangle $m^2 + m - 2$ of $2m^2 - m$ where $n = 2m$ — the authors write:

> "Determining the maximal number of simultaneously active pairs for a non-trivial configuration with $\sum_k z_k = 0$, and whether the optimal configurations for $\pi^{\max}_{n,2}$ always achieve this maximum is an interesting open question." [1], Remark 4.5

"Non-trivial" is their own term, fixed earlier in the paper: "the conjecture always holds for the trivial configuration $z_1 = \cdots = z_n = 0$, so for the remainder of this paper, we will implicitly exclude this configuration and assume $\sum_{k=1}^{n} |z_k| > 0$" [1, §3.1]. Only the all-zero configuration is excluded, and individual z_k are allowed to vanish — as they are in the class over which [1, Proposition 3.3] takes the minimum defining its own version of (2). Section 2 states both conventions and compares them; that class is what the first of the present paper's new results turns on.

What this paper settles.

(1) **The house, exactly** (Theorems 3.1 and 3.2). With the lattice coordinates $A = (0, 0)$, $B = (30, 0)$, $C = (30, 16)$, $D = (15, 36)$, $E = (0, 16)$, the five edge vectors sum to zero, the perimeter is 112, the largest two-edge triangle perimeter is 80, so $\delta = 5/7$ *exactly* and $\pi^{\max}_{5,2} \leq 5/7$; the pentagon is strictly convex, and exactly 7 of the 10 pairs are active, the three others having values $41 + \sqrt{241}$ (twice) and 32.

(2) **The 3-4 rectangle, exactly and uniformly in m** (Theorems 3.3 and 3.4). At $n = 2m$ with $m \geq 2$ the perimeter is $6m+2$, the maximum is 12, so $\delta = 4/(n+\frac{2}{3})$ exactly and $\pi^{\max}_{n,2} \leq 4/(n+\frac{2}{3})$

for *every* even $n \geq 4$; and exactly $m^2 + m - 2$ of the $2m^2 - m$ pairs are active. Both are proved for general m in one argument, not one n at a time.

(3) **The regular polygon comparison** (Theorem 4.1). The inequality

$$\frac{4}{x + \frac{2}{3}} < \frac{4}{x} \cos^2\Big(\frac{\pi}{2x}\Big),$$

which [1] asserts in one clause without computing it, holds for every *real* $x \geq 5$, so the regular n-gon is never optimal for even $n \geq 6$. At the pentagon, $(4/5)\cos^2(\pi/10) = (5+\sqrt{5})/10$ exactly, and the house improves on it by $(5\sqrt{5} - 11)/14$, which pins the "approximately 1.3%" of [1] to the interval $(0.0128, 0.0129)$.

(4) **A bracket** (Theorem 3.5). Every configuration of $n \geq 2$ nonzero vectors summing to zero has $\delta(z) \geq 2/n$, so the upper bounds above are not vacuous: $2/5 \leq \pi^{\max}_{5,2} \leq 5/7$ and $1/m \leq \pi^{\max}_{2m,2} \leq 4/(2m+\frac{2}{3})$. These brackets are proved from the definitions here and are *not* the state of the art: the live window at $n = 5$ is $2/3 \leq \pi^{\max}_{5,2} \leq 5/7$, its left end being the $4/(n + 1)$ of [1, Theorem 3.8] (Section 3).

(5) **The answer at** $n = 4$ (Theorem 5.2). The square is, by [1, Theorem 3.16] at $n = 4$, the unique optimal configuration for $\pi^{\max}_{4,2} = (2 + \sqrt{2})/4$, and it activates 4 of its 6 pairs; the non-trivial zero-sum configuration $(1, -1, 0, 0)$ activates 5. So at $n = 4$ an optimal configuration does *not* achieve the maximal number of simultaneously active pairs, and the second half of the question quoted above has a negative answer there. The phenomenon is confined to $n = 4$: the same degenerate family ties the house at $n = 5$ and loses at $n = 6$ and $n = 8$ (Proposition 5.3).

(6) **The exact maximum for** $n \leq 4$ (Section 6). At $n = 3$ every zero-sum triple has pmax = mass and all three pairs active (Theorem 6.1). At $n = 4$ the maximum over the non-trivial class is exactly 5, and any configuration attaining it is a permutation of $(u, -u, 0, 0)$ with $u \neq 0$ (Theorem 6.2); over the configurations with every $z_k \neq 0$ it is exactly 4, attained by the square and by the 3-4 rectangle (Theorem 6.3). So the first half of the question of [1, Remark 4.5] is settled at $n = 4$ in both readings, and item (5) sharpens: at $n = 4$ the maximum is attained *only* outside the class of genuine quadrilaterals.

(7) **A square-root-free criterion, and a closure bound** (Section 7). The identity $(M - |a| - |b|)^2 - |a+b|^2 = (M-2|a|)(M-2|b|) - 2\big(|a||b| + \langle a, b\rangle\big)$ turns activity into a polynomial condition (Theorem 7.1); summing it over all pairs gives an exact deficit identity, hence $\operatorname{mass}(z) \leq c_n \operatorname{pmax}(z)$ with $c_n = (n-1) - \sqrt{(n-1)(n-2)/2}$ for *every* list of $n \geq 2$ complex numbers, with equality exactly when all $\binom{n}{2}$ pairs are active (Theorem 7.2).

(8) **Degenerate configurations, at every** n (Theorem 8.1). Call an entry *saturated* when $2|z_k| = \operatorname{pmax}(z)$. A zero-sum configuration with a nonzero saturated entry is collinear, has mass = pmax and hence $\delta = 1$, and has at most $2n - 3$ active pairs; over that class the maximum is exactly $2n - 3$ for every $n \geq 3$, attained by $(1, -1, 0, \ldots, 0)$. This is the first upper bound on act in this paper that holds at every n.

(9) **A vector model and two structure lemmas** (Section 9). When no entry is saturated, each entry z has a plane vector representative W with $(\operatorname{pmax}(z) - 2|z|)W^2 = 2z$, in which a pair is active exactly when $\langle W_i, W_j\rangle^2 = 1$ (Theorem 9.1). Two consequences hold at every n with no convexity, ordering or zero-sum hypothesis: an entry with $4|w| < \operatorname{pmax}(z)$ is active with at most two entries (Theorem 9.2), and an entry lying strictly inside the cone spanned by an active non-parallel pair satisfies $4|z_u| < \operatorname{pmax}(z)$ (Theorem 9.3).

(10) **An odd-n family** (Section 10). For each $g \geq 2$ there is an explicit configuration $T(g)$ of $n = 2g + 1$ nonzero vectors summing to zero with pmax = 4 and exactly $g^2 + g$ active pairs (Theorem 10.1); since $g^2 + g > 2n - 3$ for $g \geq 3$, the degenerate family of item (5) is not the odd-n maximiser. Its δ gives $\pi^{\max}_{n,2} < (4/n)\cos^2(\pi/2n)$ for every odd $n \geq 7$ and $\pi^{\max}_{7,2} \leq (15 + \sqrt{57})/42 = 0.5369\ldots$ against the regular heptagon's $0.5431\ldots$ (Theorem 10.2), where [1] exhibits no configuration at all.

The results offered as new are those of items (5), (6), (8) and (10) together with Theorems 9.2 and 9.3 of item (9); items (1)–(4) reproduce or make exact what [1] prints, and item (7) and Theorem 9.1 are tools — a polynomial identity, a summation, a half-angle square root — that nobody would call new mathematics. Section 13 says which is which and what search was run. Nothing here touches the optimality halves of Conjectures 4.3 and 4.4; Section 12 says precisely why not.

2. The problem in coordinates

Definition 2.1 (configurations)**.** Fix $n \geq 1$. A tuple $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is

- *non-trivial* when $\sum_k z_k = 0$ and $z \neq 0$ — the term of [1, §3.1];
- *admissible* when $\sum_k z_k = 0$ and $z_k \neq 0$ for every k.

Admissible is strictly stronger, and the two differ exactly on the configurations with some vanishing entry. Throughout, $\pi^{\max}_{n,2}$ is the infimum (2) of δ over the *admissible* configurations of size n.

Which class [1] **uses.** Definition 2.1 is stricter than the source's own convention, and we say so explicitly because the gap between the two classes is the subject of Section 5. [1, Proposition 3.3] defines, for $2 \leq d \leq n$,

$$\pi^{\max}_{n,d} := \min\Big\{ \max_{S \in \binom{[n]}{d}} \Big(\sum_{s\in S} |z_s| + \Big| \sum_{s \in S} z_s \Big| \Big) \ : \ z \in \mathbb{C}^n,\ \sum_k |z_k| = 1,\ \sum_k z_k = 0 \Big\},$$

and its feasible set — written $\mathcal{Z}_n$ in the proof of [1, Theorem 3.16] — puts *no* condition on the individual z_k: the normalisation $\sum_k |z_k| = 1$ excludes the all-zero tuple and nothing else. At $d = 2$ the objective is $\mathrm{pmax}(z)$ and the normalisation makes $\mathrm{mass}(z) = 1$, so the source's quantity is the infimum of δ over the *non-trivial* configurations. The source calls it $\pi^{\max}_{n,2}$, which is also our symbol for (2); for this one comparison write $\pi^{\mathrm{nt}}_{n,2}$ for the source's. It runs over the larger class, so

$$\pi^{\max}_{n,2} \ \geq \ \pi^{\mathrm{nt}}_{n,2},$$

with equality at every n at which some *admissible* configuration attains $\pi^{\mathrm{nt}}_{n,2}$. At $n = 4$ the square does, so $\pi^{\max}_{4,2} = \pi^{\mathrm{nt}}_{4,2} = (2 + \sqrt{2})/4$ (Theorem 5.1 with [1, Corollary 2.7, Proposition 3.3]). That equality at $n = 4$ is the only identification of the two made anywhere below, and it is what lets Section 5 write $\pi^{\max}_{4,2}$ for the quantity whose optimum [1] classifies. At every other n, $\pi^{\max}_{n,2}$ carries only bounds that are certified by an admissible configuration, and so hold of $\pi^{\mathrm{nt}}_{n,2}$ as well, or that are stated for $\pi^{\max}_{n,2}$ alone (Theorem 3.5). Section 4 of [1] restates the same quantity in polygon language, as $\min\{\delta(\mathcal{X}) : \mathcal{X}$ a "non-trivial convex n-gon"$\}$ with δ introduced "for a nondegenerate polygon" [1, Problem 4.1]; neither quoted term is defined there, and in polygon language a vanishing edge vector is hard to admit, so that phrasing pulls the other way. We take the displayed formula, which is unambiguous, to be the source's convention.

Two remarks on Definition 2.1 keep the statements below aligned with [1]. First, admissibility does *not* ask that consecutive edge vectors be non-parallel. It cannot: the 3-4 rectangle of [1, Conjecture 4.4] has $2m$ edge vectors taking only four distinct values, so as a polygon it is a $3(m-1) \times 4$ rectangle whose two long sides are each subdivided into $m - 1$ edges of length 3. Requiring non-parallel consecutive edges would exclude that paper's own conjectured optimum. Second, every configuration exhibited below is admissible, hence non-trivial; since both infima run over classes containing it, each of them certifies the *same* upper bound for $\pi^{\max}_{n,2}$ and for $\pi^{\max}_{n,2}$, and the upper bounds of Sections 3 and 4 are therefore insensitive to which class one takes. The distinction matters in exactly one place, Section 5, and there it is the point.

The two configurations of [1] are *lattice* configurations: their entries are Gaussian integers, and — this is the arithmetic accident the whole exact treatment runs on — every $|z_k|$ and almost every $|z_i + z_j|$ occurring in them is an integer. We write $\mathrm{pt}(a, b) = a + bi$ and record the evaluation used throughout: if $a^2 + b^2 = c^2$ with $c \geq 0$ then $|\mathrm{pt}(a, b)| = c$, so that a value of pv at two lattice points is determined by three Pythagorean identities over $\mathbb{Z}$.

Finally, for orientation, put $\mathrm{cross}(a, b) = \mathrm{Re}(a)\,\mathrm{Im}(b) - \mathrm{Im}(a)\,\mathrm{Re}(b)$, twice the signed area of the triangle spanned by a and b, and $\langle a, b\rangle = \mathrm{Re}(a)\,\mathrm{Re}(b) + \mathrm{Im}(a)\,\mathrm{Im}(b)$, the real inner product of $\mathbb{C} = \mathbb{R}^2$. All consecutive $\mathrm{cross}(z_k, z_{k+1})$ being positive says that the closed path turns strictly left at every vertex, i.e. that it is a strictly convex polygon traversed counterclockwise. The two are tied by the Lagrange identity $\langle a,b\rangle^2 + \operatorname{cross}(a,b)^2 = |a|^2|b|^2$, which is the workhorse of Sections 6–9.

3. The two conjecturally optimal configurations, exactly

3.1. The house.

Theorem 3.1 (the house)**.** *Let* $H = (u_1, \dots, u_5)$ *with*

$$u_1 = 30, \quad u_2 = 16i, \quad u_3 = -15 + 20i, \quad u_4 = -15 - 20i, \quad u_5 = -16i,$$

the edge vectors of the pentagon $A = (0,0)$*,* $B = (30,0)$*,* $C = (30,16)$*,* $D = (15,36)$*,* $E = (0,16)$*. Then* H *is an admissible configuration of size* 5*; its side lengths are* $30, 16, 25, 25, 16$*, so* $\operatorname{mass}(H) = 112$*; and* $\operatorname{pmax}(H) = 80$*. Consequently*

$$\delta(H) = \frac{80}{112} = \frac{5}{7} \quad \textit{exactly,} \qquad \textit{and} \qquad \pi^{\max}_{5,2} \le \frac{5}{7}.$$

Moreover H *is strictly convex and counterclockwise:* $\operatorname{cross}(u_1,u_2) = 480$*,* $\operatorname{cross}(u_2,u_3) = 240$*,* $\operatorname{cross}(u_3,u_4) = 600$*,* $\operatorname{cross}(u_4,u_5) = 240$ *and* $\operatorname{cross}(u_5,u_1) = 480$*.*

The value 5/7 is [1, Conjecture 4.3]; what is proved here is the identity, and with it the unconditional upper bound. The coordinates are not printed in [1], which describes the house verbally — “$ABCE$ is a rectangle and $|AB| = 30$, $|BC| = |EA| = 16$, $|CD| = |DE| = 25$” — and otherwise only draws it. The description already fixes the shape up to similarity: the rectangle $ABCE$ is 30×16, and the roof CDE has base $CE = 30$ and equal sides 25, so its height is $\sqrt{625 - 225} = 20$, the 15-20-25 triangle. What it does not fix is the placement, and for that we read the source’s Figure 3a — not from the printed page but from the vector graphics of the version 1 e-print. The stroked path of its `house.pdf` has the five vertices, in the drawn cyclic order and in PostScript points,

$$(-85.039, -102.047),\ (85.039, -102.047),\ (85.039, -11.339),\ (0, 102.047),\ (-85.039, -11.339),$$

which at 5.6693 points per unit is exactly A, B, C, D, E as above: apex over the midpoint of AB, traversed counterclockwise, neither reflected nor otherwise placed. The companion Figure 3b draws $u_1, \dots, u_5$ from a common origin, and on the same scale its five arrows are $(30,0)$, $(0,16)$, $(-15,20)$, $(-15,-20)$, $(0,-16)$ — the tuple H itself.

Proof sketch. Each of the five side lengths is a Pythagorean evaluation over $\mathbb{Z}$ ($30^2+0^2 = 30^2$, $0^2+16^2 = 16^2$, $15^2 + 20^2 = 25^2$ twice, $0^2 + 16^2 = 16^2$), and their sum is 112. For pmax, all ten values of pv are computed in Theorem 3.2; seven of them are 80 and the other three are smaller, the comparison resting on the single irrationality $\sqrt{241} < 39$. Convexity is five integer determinants. All of this is integer arithmetic on ten integers; no search of any kind is involved. □

Theorem 3.2 (the ten pair values)**.** *In lexicographic order of the index pair, the ten values* $\operatorname{pv}(u_i, u_j)$ *of the house are*

$$80,\ 80,\ 80,\ 80,\ 80,\ 41 + \sqrt{241},\ 32,\ 80,\ 41 + \sqrt{241},\ 80.$$

Exactly seven of them equal $\operatorname{pmax}(H) = 80$*, so* $\operatorname{act}(H) = 7$ *out of* $\binom{5}{2} = 10$ *pairs. The three inactive pairs are* $\{u_2, u_4\}$ *and* $\{u_3, u_5\}$*, of value* $41 + \sqrt{241} \approx 56.52$*, and* $\{u_2, u_5\}$*, of value* 32*.*

The count 7 of 10 is the one [1, Remark 4.5] prints; the full list is more than the count, and it isolates what the count depends on. Table 1 records the arithmetic. Thirteen of the fifteen moduli involved — five side lengths and ten pairwise sums — are integers; the exceptions are the two sums $u_2 + u_4 = -15 - 4i$ and $u_3 + u_5 = -15 + 4i$, both of modulus $\sqrt{241}$. The whole separation of active from inactive pairs is the inequality $\sqrt{241} < 39$, i.e. $241 < 1521$.

pair	$\lvert u_i\rvert + \lvert u_j\rvert + \lvert u_i + u_j\rvert$	value	pair	$\lvert u_i\rvert + \lvert u_j\rvert + \lvert u_i + u_j\rvert$	value
$\{1,2\}$	$30+16+34$	80	$\{2,4\}$	$16+25+\sqrt{241}$	≈ 56.52
$\{1,3\}$	$30+25+25$	80	$\{2,5\}$	$16+16+0$	32
$\{1,4\}$	$30+25+25$	80	$\{3,4\}$	$25+25+30$	80
$\{1,5\}$	$30+16+34$	80	$\{3,5\}$	$25+16+\sqrt{241}$	≈ 56.52
$\{2,3\}$	$16+25+39$	80	$\{4,5\}$	$25+16+39$	80

TABLE 1. The ten two-edge triangle perimeters of the house. Seven equal 80; the polygon perimeter is 112, so $\delta = 5/7$.

3.2. **The 3-4 rectangle.**

Theorem 3.3 (the 3-4 rectangle)**.** *For $m \geq 1$ let R_m be the configuration of size $2m$ consisting of $m-1$ entries equal to 3, then one entry $4i$, then $m-1$ entries equal to -3, then one entry $-4i$. Then R_m is admissible,* $\operatorname{mass}(R_m) = 6m+2$*, and* $\operatorname{pmax}(R_m) = 12$ *for every $m \geq 2$. Consequently, writing $n = 2m$,*

$$\delta(R_m) = \frac{12}{6m+2} = \frac{4}{n+\frac{2}{3}} \quad \textit{exactly,} \qquad \textit{and} \qquad \pi^{\max}_{n,2} \leq \frac{4}{n+\frac{2}{3}} \quad \textit{for every even } n \geq 4.$$

Three things are worth separating out. The value $4/(n+\frac{2}{3})$ is [1, Conjecture 4.4], stated there for even $n \geq 6$; the bound above also holds at $n = 4$, where it is *not* sharp, since $\pi^{\max}_{4,2} = (2+\sqrt{2})/4 \approx 0.85355$ is below $4/(4+\frac{2}{3}) = 6/7 \approx 0.85714$ (Section 2). Admissibility is part of the statement and is not automatic: the $2m$ entries take only four distinct values, so as a polygon R_m is a degenerate $2m$-gon (Section 2). And the bound is proved once for all m, not m by m.

Proof sketch. The whole content is the 3-4-5 triangle: $|3|+|4i|+|3+4i| = 3+4+5 = 12$, which is also $|3|+|3|+|3+3| = 12$. Only four values occur among the entries, so only ten unordered value pairs can occur at all, of which the eight realisable at *distinct* positions are

$$(3,3):12, \quad (3,4i):12, \quad (3,-3):6, \quad (3,-4i):12,$$
$$(-3,-3):12, \quad (4i,-3):12, \quad (4i,-4i):8, \quad (-3,-4i):12.$$

Each is again a Pythagorean evaluation over $\mathbb{Z}$. The one delicate point is that $(4i,4i)$, worth $16 > 12$, does not occur — and it fails to occur only because $4i$ appears once; the argument is a pairwise statement over the concatenation $3^{m-1}\,4i\,(-3)^{m-1}\,(-4i)$ that keeps this visible rather than hiding it in a case count. Attainment is the pair $(3,4i)$, present as soon as $m \geq 2$. The mass is $3(m-1)+4+3(m-1)+4 = 6m+2$. ☐

Theorem 3.4 (the active count of the 3-4 rectangle)**.** *For every $m \geq 1$ the configuration R_m has $\binom{2m}{2} = 2m^2 - m$ pairs, and for every $m \geq 2$ exactly m^2+m-2 of them are active. The inactive pairs are the $(m-1)^2$ pairs $\{3,-3\}$, of value 6, together with the single pair $\{4i,-4i\}$, of value 8.*

Proof sketch. The count is $2\binom{m-1}{2} + 4(m-1) = (m-1)(m+2) = m^2+m-2$, and it is obtained uniformly in m from two elementary lemmas about counting over unordered pairs of a list, proved here for this purpose: the number of pairs satisfying a predicate splits over a concatenation $A +\!\!+\, B$ into the pairs inside A, the pairs inside B, and the $|A|\,|B|$ crossing pairs; and a constant block of length n contributes $\binom{n}{2}$ or 0 according to whether the predicate holds of that constant pair. Equivalently, the complement is counted: $2m^2 - m - (m-1)^2 - 1 = m^2+m-2$. No value of m is enumerated. ☐

3.3. **A bracket.** Every statement so far is an upper bound on an infimum, which says nothing at all unless the infimum is known to be positive. The following makes the brackets non-empty without importing the sharper lower bound of [1].

Theorem 3.5 (a uniform lower bound)**.** *Let z be an admissible configuration of size $n \geq 2$. Then $\delta(z) \geq 2/n$. Consequently $2/n \leq \pi^{\max}_{n,2}$ whenever an admissible configuration of size n exists, and in* *particular*

$$\frac{2}{5} \le \pi^{\max}_{5,2} \le \frac{5}{7}, \qquad \frac{1}{m} \le \pi^{\max}_{2m,2} \le \frac{4}{2m+\frac{2}{3}} \quad (m \ge 2).$$

Proof sketch. From $|a+b| \ge 0$ we get $\mathrm{pv}(a,b) \ge |a|+|b|$. Summing over all $\binom{n}{2}$ pairs, each index appears in $n-1$ of them, so $\sum_{i<j} \mathrm{pv}(z_i,z_j) \ge (n-1)\,\mathrm{mass}(z)$. The maximum of $\binom{n}{2}$ numbers is at least their average, so $\mathrm{pmax}(z) \ge \frac{n-1}{\binom{n}{2}}\,\mathrm{mass}(z) = \frac{2}{n}\,\mathrm{mass}(z)$, and $\mathrm{mass}(z) > 0$ because some entry is nonzero. □

The bound $2/n$ is deliberately weak. [1] proves the much better $\pi^{\max}_{n,2} \ge \mu^{\max}_{n,2} = 4/(n+1)$ by an appendix argument (its Appendix B) that is not reproduced here; $2/n$ is a cheap independent certificate that the quantity is bounded away from zero, and that is all it is asked to be. The normalisations agree: its Theorem 3.8 reads, for $n \ge 3$ and *every* $z \in \mathbb{C}^n$, $\max_{1\le k<l\le n}(|z_k|+|z_l|+|z_k+z_l|) \ge \frac{4}{n+1}\sum_k |z_k|$, which in the notation (1) is $\mathrm{pmax}(z) \ge \frac{4}{n+1}\,\mathrm{mass}(z)$, i.e. $\delta(z) \ge 4/(n+1)$ verbatim. That it is $\mu^{\max}_{n,2}$ that this value names, and that $\pi^{\max}_{n,2} \ge \mu^{\max}_{n,2}$, are its Remarks 3.6 and 3.4.

Two consequences of that comparison should be stated plainly, because the brackets of Theorem 3.5 are easy to mistake for the state of the art and they are not. First, the *live* window at $n = 5$ is

$$\frac{2}{3} \;\le\; \pi^{\max}_{5,2} \;\le\; \frac{5}{7},$$

of width $1/21 \approx 0.0476$, its left end being $4/(n+1)$ at $n = 5$ and not the $2/5$ above; it is that window, not $[2/5, 5/7]$, that a proof of [1, Conjecture 4.3] has to close, and any lower bound below $2/3$ — including Theorem 3.5, including the $1/c_5 = (4+\sqrt{6})/10 \approx 0.6449$ of Theorem 7.2 below, and including anything the modulus relaxation of Remark 7.3 can reach — proves nothing new about the value. Second, what Theorem 3.5 and Theorem 7.2 *are* good for is independence and generality: they are proved here from the definitions, in two and in one line respectively, so that no upper bound in this paper rests on an argument reproduced from elsewhere, and Theorem 7.2 holds with no hypothesis on z at all.

4. The regular n-gon is never optimal

[1, Proposition 4.2] records the value of δ at the regular n-gon, $\delta(A_n) = (4/n)\cos^2(\pi/2n)$, and then disposes of it in a clause: "Since $(4/n)\cos^2(\pi/2n) > 4/(n+2/3)$ (similar to an inequality in the proof of Theorem 3.17), the above 3-4 *rectangle* configuration shows that when $d = 2$ and $n \ge 6$ is even, the optimal polygon is never the regular n-gon." For the pentagon it says only: "We conjecture that for $n = 5$ the optimal polygon is approximately 1.3% better than the regular pentagon." Neither number is computed there. Both are below. We take the identity $\delta(A_n) = (4/n)\cos^2(\pi/2n)$ from [1] and do not reprove it; what follows is everything downstream of that value.

Theorem 4.1 (the comparison, and the pentagon exactly). (1) *For every real* $x \ge 5$, $\dfrac{4}{x+\frac{2}{3}} < \dfrac{4}{x}\cos^2\!\Big(\dfrac{\pi}{2x}\Big)$.

(2) *Hence for every even* $n = 2m \ge 6$, $\pi^{\max}_{n,2} < \dfrac{4}{n}\cos^2\!\Big(\dfrac{\pi}{2n}\Big) = \delta(A_n)$*: the regular n-gon is never optimal.*

(3) *At the pentagon,* $\dfrac{4}{5}\cos^2\!\Big(\dfrac{\pi}{10}\Big) = \dfrac{5+\sqrt{5}}{10}$ *exactly, and* $\pi^{\max}_{5,2} < \dfrac{5+\sqrt{5}}{10}$.

(4) *The house improves on the regular pentagon by exactly*

$$1 - \frac{5/7}{(5+\sqrt{5})/10} = \frac{5\sqrt{5}-11}{14}, \qquad \textit{and} \qquad 0.0128 < \frac{5\sqrt{5}-11}{14} < 0.0129.$$

Proof sketch. (1) is the sketch of [1] made exact. Write $t = \pi/x$. The half-angle identity gives $\cos^2(\pi/2x) = \frac{1}{2} + \frac{1}{2}\cos t$, so the claim is $1 - \cos t < 4/(3x+2)$. Combine $1 - \cos t \le t^2/2$ with $t^2/2 < 4/(3x+2)$, the latter being $\pi^2(3x+2) < 8x^2$, which follows from $\pi < 3.15$ and $x \ge 5$ by a quadratic estimate. (2) is (1) at $x = n$ together with Theorem 3.3. (3) needs no bound on π: the exact value $\cos(\pi/5) = (1+\sqrt{5})/4$ gives $(4/5)\cos^2(\pi/10) = \frac{2}{5}(1+\cos(\pi/5)) = (5+\sqrt{5})/10$, and then

$5/7 < (5+\sqrt{5})/10$ is $225 < 245$ after clearing. (4) is elementary algebra in $\sqrt{5}$, and the numerical bracket follows from $2.236 < \sqrt{5} < 2.2361$. □

Part (1) is stated for every real $x \geq 5$ and not merely for even integers $n \geq 6$, and the hypothesis $x \geq 5$ is not decoration: at $x = 4$ the inequality is *false*, because $(4/4)\cos^2(\pi/8) = (2+\sqrt{2})/4 \approx 0.85355$ is below $4/(4+\frac{2}{3}) = 6/7 \approx 0.85714$. That is exactly why [1] states Conjecture 4.4 for $n \geq 6$ and not for $n \geq 4$, and it is recorded as a control in Section 11. Part (4) turns the "approximately 1.3%" of [1] into an algebraic number: $(5\sqrt{5}-11)/14 = 0.012881\ldots$

Odd n is not touched by Theorem 4.1(2), and [1] leaves it open; it is settled, from $n = 7$ on, by the explicit family of Section 10.

5. Active pairs, and the answer at $n = 4$

5.1. Two more configurations.

Theorem 5.1 (the collapsed pair, and the square)**.** *For $n \geq 2$ let $C_n = (1, -1, 0, \ldots, 0)$, with $n-2$ zero entries. Then C_n is a non-trivial zero-sum configuration — but not an admissible one — with* $\mathrm{pmax}(C_n) = 2$ *and*

$$\mathrm{act}(C_n) = \binom{n}{2} - \binom{n-2}{2} = 2n-3.$$

Let $S = (1, i, -1, -i)$ be the unit square. Then S is admissible of size 4, $\delta(S) = (2+\sqrt{2})/4$, hence $\pi^{\max}_{4,2} \leq (2+\sqrt{2})/4$, and $\mathrm{act}(S) = 4$ *of its 6 pairs.*

Proof sketch. For C_n every pair value is 2 — $\mathrm{pv}(1,-1) = 2$ and $\mathrm{pv}(\pm 1, 0) = 2$ — except the pairs of two zeros, where it is 0; so the maximum is 2 and the active pairs are exactly those meeting a nonzero entry, of which there are $\binom{n}{2} - \binom{n-2}{2} = 2n-3$. The count uses the same two counting lemmas as Theorem 3.4. For S, four of the six pair values are $2+\sqrt{2}$ (the pairs of perpendicular edges) and two are 2 (the pairs of opposite edges), and $\mathrm{mass}(S) = 4$. The value $(2+\sqrt{2})/4$ is the one [1] records for $\pi^{\max}_{4,2}$ (its Corollary 2.7 and Proposition 3.3); here it appears only as an upper bound, the matching lower bound being that paper's and not reproved. □

The configuration C_n is degenerate as a polygon — a doubly traversed segment — and does not appear in [1], which considers only configurations it expects to be optimal. It is admitted here for one reason: it satisfies the definition of *non-trivial* that the open question of [1, Remark 4.5] is posed with, and it lies in the feasible set of [1, Proposition 3.3].

5.2. An optimum that is not active-maximal.

Theorem 5.2 (an optimum that is not active-maximal)**.** *At $n = 4$,*

$$\mathrm{act}(S) = 4, \qquad \mathrm{act}(C_4) = 5, \qquad \mathrm{act}(S) < \mathrm{act}(C_4).$$

Consequently, since [1, Theorem 3.16] *at $n = 4$ (where $d = n-2 = 2$) shows that the square is the* unique *optimal configuration for $\pi^{\max}_{4,2}$, an optimal configuration for $\pi^{\max}_{4,2}$ does not achieve the maximal number of simultaneously active pairs over non-trivial zero-sum configurations of size* 4*. The second half of the open question of* [1, Remark 4.5] *has a negative answer at $n = 4$.*

Proof. The two counts are Theorem 5.1 at $n = 4$: $2 \cdot 4 - 3 = 5$ and $\mathrm{act}(S) = 4$. The configuration $C_4 = (1, -1, 0, 0)$ is non-trivial in the sense of [1, §3.1], so the maximum in the question is at least 5, while the optimal configuration attains 4. □

Three things about Theorem 5.2 are worth stating plainly, because each of them is a place where the claim could be read as wider than it is.

First, the *uniqueness* of the square as the optimum at $n = 4$ is quoted from [1] and is not reproved here; what is proved here is the pair of counts and their strict inequality, which is what settles the question once that uniqueness is granted. In fact uniqueness is more than is needed: it suffices that the square be *an* optimum. The statement being used is [1, Theorem 3.16] — "For $n \geq 4$, the vertices of a regular n-gon centered about the origin are the only optimal configurations for $\pi^{\max}_{n,n-2}$" — taken at

$n = 4$, where $n - 2 = 2$. Three features of it matter here and all are in our favour. It optimises over the same feasible set $\mathcal{Z}_n$ as [1, Proposition 3.3], so the uniqueness holds over the *non-trivial* class and a fortiori over the admissible one. Its description of the optimum is up to rotation and up to permutation of the entries; at $n = 4$ they are the tuples cS, $c \in \mathbb{C} \setminus \{0\}$, and their reorderings, with the normalisation $\sum_k |z_k| = 1$ pinning $|c| = 1/4$. Since pv is homogeneous and rotation-invariant and act is unchanged by reordering, act is the same at every one of them, namely 4. And it is a statement about π, not about μ: the classification of the optimal configurations for $\mu^{\max}_{n,2}$ is a different theorem of [1] (its Theorem 3.12), and it is not the one used here.

Second, the answer depends on which class *non-trivial* names, and the dependence is not a technicality. Under the definition of [1, §3.1], which excludes only the all-zero configuration — and which is the class over which [1, Proposition 3.3] takes the minimum defining $\pi^{\mathrm{nt}}_{n,2}$ in the first place — the answer at $n = 4$ is *no*, as above. Under the stricter reading in which every z_k is required to be nonzero — what admissibility asks, and what a genuine 4-gon needs — the configuration C_4 is no longer a competitor, and the answer at $n = 4$ turns into *yes*; Section 6 proves both readings exactly.

Third, $n = 4$ is an anomaly and not the beginning of a pattern.

Proposition 5.3 (the phenomenon is confined to $n = 4$)**.**

$$\operatorname{act}(C_4) = 5 > 4 = \operatorname{act}(R_2), \qquad \operatorname{act}(C_5) = 7 = \operatorname{act}(H),$$
$$\operatorname{act}(C_6) = 9 < 10 = \operatorname{act}(R_3), \qquad \operatorname{act}(C_8) = 13 < 18 = \operatorname{act}(R_4).$$

Remark 5.4. Comparing the two closed forms of Theorems 5.1 and 3.4 explains the table: at $n = 2m$ the degenerate count $2n - 3 = 4m - 3$ exceeds $m^2 + m - 2$ exactly when $m^2 - 3m + 1 < 0$, that is only for $m = 2$. This one-line consequence of the two theorems is not itself part of the formal development, which certifies the four numerical comparisons above. At *odd* n the comparison of Proposition 5.3 is genuinely different, and Theorem 10.1 shows that C_n stops being the best known configuration there from $n = 7$ on.

Remark 5.5 (a floating-point search, outside the formal development). For orientation only, and with a different standing from every numbered statement in this paper: for each target set T of pairs, a least-squares system asking $\mathrm{pv}_{ij} = M$ for $(i,j) \in T$ together with $\sum_k z_k = 0$, $\sum_k |z_k| = 1$ and one-sided penalties off T was solved from 12 random starts, a residual below 10^{-20} counting as achievable; about 50 CPU-minutes in total. At $n = 4$, 6 pairs were never achieved and 5 only with two vanishing entries; at $n = 5$, 10, 9 and 8 were never achieved over all 1, 10 and 45 target sets, and 7 was; at $n = 6$, $15, 14, 13, 12, 11$ were never achieved over all 1, 15, 105, 455 and 1365 target sets, and 10 was — at a configuration the search found on its own with moduli in ratio 4:3:3:3:3:4, which is the 3-4 rectangle. Every negative here means “not found by this search”, never “impossible”; a target set is declared unachievable when 12 restarts fail, which can be a failure of the search rather than an obstruction. The $n = 4$ negatives are now theorems (Section 6); the $n = 5$ and $n = 6$ ones are not. Taken together the search says that the expectation of [1] — that an optimal configuration activates the maximum number of pairs — appears to hold at $n = 5$ and $n = 6$, and that $n = 4$ is the exception.

6. The exact maximum for $n \leq 4$

This section settles the *first* half of the question of [1, Remark 4.5] — the value of the maximum — for $n \leq 4$, in both readings of *non-trivial*, and in each case the maximum is attained, so the answers are exact rather than bounds.

Theorem 6.1 ($n = 3$: everything is active)**.** *Let* $z_1 + z_2 + z_3 = 0$*. Then* $\operatorname{pmax}(z) = \operatorname{mass}(z)$ *and all* $\binom{3}{2} = 3$ *pairs are active; in particular* $\delta(z) = 1$ *whenever* $z \neq 0$*. No non-triviality hypothesis is needed.*

Proof sketch. $z_1 + z_2 = -z_3$ gives $\mathrm{pv}(z_1, z_2) = |z_1| + |z_2| + |z_3| = \operatorname{mass}(z)$, and the same for the other two pairs; three equal numbers have that common value as their maximum. □

So the question has no content below $n = 4$, and $n = 4$ is the first size at which $\binom{n}{2}$ is *not* attained.

Theorem 6.2 (the maximum at $n = 4$, non-trivial class)**.** *Let z be a non-trivial zero-sum configuration of size 4. Then*

$$\operatorname{act}(z) \le 5,$$

the value 5 is attained at $C_4 = (1, -1, 0, 0)$, and if $\operatorname{act}(z) = 5$ then z is a permutation of $(u, -u, 0, 0)$ for some $u \ne 0$. So the maximum over the class of [1, §3.1] *is exactly 5, and a configuration attains it only by having two vanishing entries.*

Theorem 6.3 (the maximum at $n = 4$, admissible class)**.** *Let z be an admissible configuration of size 4. Then $\operatorname{act}(z) \le 4$, and the value 4 is attained both by the square S and by the 3-4 rectangle R_2. So the maximum over the admissible class is exactly 4.*

Together with Theorem 5.2 this completes $n = 4$: the maximum over the source's own class is 5 and the optimal configuration attains 4, so the answer to the second half of the question is negative there; and the reason is structural rather than accidental, since by Theorem 6.2 the maximisers are *exactly* the configurations that fail to be quadrilaterals at all. Under the stricter reading the two halves agree: the maximum is 4 and the square attains it.

Proof sketch of Theorems 6.2 and 6.3. Write $a_k = |z_k|$. The zero-sum relation pairs the six pair values two by two: $|z_3 + z_4| = |z_1 + z_2|$, $|z_2 + z_4| = |z_1 + z_3|$, $|z_2 + z_3| = |z_1 + z_4|$, each one $\pm$-sign away from $z_1 + z_2 + z_3 + z_4 = 0$. Hence "two opposite pairs are both active" is a *linear* condition on $a_1, \dots, a_4$, and that is what makes the analysis elementary.

All six active is impossible except at $z = 0$. The three linear relations give $a_1 = a_2 = a_3 = a_4$, hence $|z_1 + z_2| = |z_1 + z_3| = |z_1 + z_4|$, hence $\langle z_1, z_2\rangle = \langle z_1, z_3\rangle = \langle z_1, z_4\rangle$. Pairing $z_2 + z_3 + z_4 = -z_1$ against z_1 gives $3\langle z_1, z_2\rangle = -a_1^2$; pairing it crosswise gives $\operatorname{cross}(z_1, z_2) + \operatorname{cross}(z_1, z_3) + \operatorname{cross}(z_1, z_4) = 0$, and by the Lagrange identity those three numbers have the same square $K = a_1^4 - \langle z_1, z_2\rangle^2$. Three reals with equal squares summing to zero force $K = 0$, so $\langle z_1, z_2\rangle^2 = a_1^4$ while $3\langle z_1, z_2\rangle = -a_1^2$, i.e. $9a_1^4 = a_1^4$ and $z = 0$. *Exactly five active forces the collapsed pair.* Say $\{3, 4\}$ is the inactive pair. The two opposite-pair relations give $a_1 = a_2 =: s$ and $a_3 = a_4 =: t$, and comparing the active pair $\{1, 2\}$ with the inactive $\{3, 4\}$ gives $t < s$, so $s > 0$. From $\{1, 3\}$ and $\{1, 4\}$ active with $a_3 = a_4$ we get $|z_1 + z_3| = |z_1 + z_4|$, i.e. $\langle z_1, z_3 - z_4\rangle = 0$, and likewise $\langle z_2, z_3 - z_4\rangle = 0$. If $z_3 \ne z_4$ then z_1, z_2 are both orthogonal to a nonzero vector of $\mathbb{R}^2$, hence parallel, and Lagrange gives $\langle z_1, z_2\rangle = \pm s^2$: the sign $+$ makes $z_1 = z_2$ and $|z_3 + z_4| = 2s > 2t$, absurd; the sign $-$ makes $z_2 = -z_1$, $z_4 = -z_3$, $\langle z_1, z_3\rangle = 0$ and then $s^2 + t^2 = (s - t)^2$, i.e. $st = 0$, absurd. If $z_3 = z_4 =: c$ then $|z_1 + c| = s + t$ is equality in the triangle inequality, which in the scale-free form $|c|\, z_1 = |z_1|\, c$ (and the same for z_2) gives $z_1 = z_2$ when $c \ne 0$, hence $2z_1 = -2c$ and $s = t$, absurd. So $c = 0$, $z_3 = z_4 = 0$ and $z_2 = -z_1 \ne 0$. Theorem 6.3 follows: five active pairs would force two vanishing entries, which admissibility forbids, and attainment is Theorems 5.1 and 3.4 at $m = 2$. □

Two features of that argument are worth naming, because they bear on what the machine checking does and does not certify. *No search of any kind occurs.* The only finite enumeration is a case split over the six labellings of the inactive pair, each an instance of one lemma about four named complex numbers with its arguments permuted, together with the observation that "at least five of six booleans are true" is an explicit disjunction of six cases. Nothing is decided by evaluation. *The classification is part of the statement.* "Up to permutation" is not a remark added after the fact: the conclusion of Theorem 6.2 is that the list z is a permutation of $[u, -u, 0, 0]$, and the permutation is exhibited in each of the six branches.

Corollary 6.4. *At $n = 3$ every zero-sum configuration has all $\binom{3}{2} = 3$ pairs active; at $n = 4$ no non-trivial zero-sum configuration has all $\binom{4}{2} = 6$ pairs active.*

7. A square-root-free criterion, and a closure bound

Every statement so far has had to fight the three square roots in $\operatorname{pv}(a, b) = |a| + |b| + |a + b|$, which is why Sections 3 and 5 work with lattice configurations, where all three moduli are integers. The next identity removes them from the *activity relation* at every n and for arbitrary complex entries. Put

$$q(a,b) = 2\big(|a|\,|b| + \langle a,b\rangle\big), \qquad P_M(a,b) = (M-2|a|)(M-2|b|).$$

By Cauchy–Schwarz $q \ge 0$ always.

Theorem 7.1 (the activity criterion)**.** *For all $a,b \in \mathbb{C}$ and all real M,*

$$\big(M-|a|-|b|\big)^2 - |a+b|^2 \;=\; P_M(a,b) - q(a,b). \tag{3}$$

If moreover $2|a| \le M$ and $2|b| \le M$, then

$$\mathrm{pv}(a,b) \le M \iff q(a,b) \le P_M(a,b), \qquad \mathrm{pv}(a,b) = M \iff q(a,b) = P_M(a,b).$$

The side condition is free: every entry of every configuration of length $n \ge 2$ satisfies $2|z_k| \le \mathrm{pmax}(z)$. Consequently, for any two entries a,b of a configuration z of length at least 2, the pair (a,b) is active if and only if $q(a,b) = P_{\mathrm{pmax}(z)}(a,b)$: activity is a polynomial condition in the coordinates and the moduli.

Proof sketch. Both sides of (3) expand to $M^2 - 2M(|a|+|b|) + 2|a||b| - 2\langle a,b\rangle$; the only input is $|a+b|^2 = |a|^2 + 2\langle a,b\rangle + |b|^2$. The equivalences follow because $2|a| \le M$ and $2|b| \le M$ force $M - |a| - |b| \ge 0$, so the inequality $|a+b| \le M - |a| - |b|$ may be squared reversibly. For the side condition, pick any partner $y \ne x$ in the list: the reverse triangle inequality gives $\mathrm{pv}(x,y) \ge |x| + |y| + \big||x| - |y|\big| \ge 2|x|$. □

No modulus of a *sum* survives on the right of (3), and that is what makes the next step possible: $\sum_{i<j} |z_i + z_j|^2$ is computable from $\sum_k |z_k|^2$ and $|\sum_k z_k|^2$, whereas $\sum_{i<j} |z_i + z_j|$ is not.

Theorem 7.2 (the deficit identity and the closure bound)**.** *Let z be any list of n complex numbers, $s = \mathrm{mass}(z)$, and let M be any real number. Then*

$$\sum_{i<j} \Big[\big(M - |z_i| - |z_j|\big)^2 - |z_i + z_j|^2\Big] = \binom{n}{2} M^2 - 2(n-1)Ms + s^2 - \Big|\sum_k z_k\Big|^2. \tag{4}$$

Consequently, for $n \ge 2$ and $c_n = (n-1) - \sqrt{(n-1)(n-2)/2}$,

$$\mathrm{mass}(z) \le c_n\ \mathrm{pmax}(z), \qquad \textit{hence} \qquad \delta(z) \ge \frac{1}{c_n} \quad \textit{whenever } \mathrm{mass}(z) > 0,$$

with no *zero-sum and no nonvanishing hypothesis. The bound is strictly stronger than Theorem 3.5 for every $n \ge 3$ and strictly weaker than the $4/(n+1)$ of* [1] *for every $n \ge 4$; the two agree at $n = 3$, where $1/c_3 = 1$. If z is zero-sum and $n \ge 2$, then* $\mathrm{mass}(z) = c_n\,\mathrm{pmax}(z)$ *holds* if and only if *all $\binom{n}{2}$ pairs are active. In particular, at $n = 4$, where $c_4 = 3 - \sqrt{3}$, the inequality is strict:* $\mathrm{mass}(z) < (3 - \sqrt{3})\,\mathrm{pmax}(z)$ *for every non-trivial zero-sum quadruple.*

Proof sketch. (4) is one induction on the list, the pair sums entering only through $\sum_{i<j} |z_i + z_j|^2 = (n-2)\sum_k |z_k|^2 + |\sum_k z_k|^2$. At $M = \mathrm{pmax}(z)$ every summand on the left is nonnegative — it is $\mathrm{pv}(z_i, z_j) \le \mathrm{pmax}(z)$, squared — and vanishes exactly on an active pair. So the right-hand side of (4) is a nonnegative quadratic in s; its smaller root is $c_n M$, and $s \le nM/2$ (Theorem 7.1, summed) puts s on that side of it. The equality clause is the vanishing of every summand. The two comparisons are proved rather than asserted: after clearing the square root, the gap in $1/c_n < 4/(n+1)$ is exactly $(n-3)^2 > 0$. Strictness at $n = 4$ is the equality clause together with $\mathrm{act}(z) \le 5 < 6$ (Theorem 6.2). □

n	3	4	5	6	8	10
$2/n$ (Theorem 3.5)	0.6667	0.5000	0.4000	0.3333	0.2500	0.2000
$1/c_n$ (Theorem 7.2)	1.0000	0.7887	0.6449	0.5442	0.4137	0.3333
$4/(n+1)$ ([1])	1.0000	0.8000	0.6667	0.5714	0.4444	0.3636

TABLE 2. Three lower bounds for δ. The first two are proved here from the definitions; the third is [1, Theorem 3.8] and is not reproved. Asymptotically $1/c_n \sim (2 + \sqrt{2})/n$.

Remark 7.3 (the modulus relaxation, and exactly where it stops)*.* The following is recorded for orientation and is not one of the numbered statements of this paper. Keep only the moduli: put $r_i = |z_i|$ and $s_{ij} = |z_i + z_j|$ at $n = 5$, keep the two identities that survive, $\sum_i r_i = 1$ and $\sum_{j \neq i} s_{ij}^2 = r_i^2 + \sum_k r_k^2$, together with $r, s \geq 0$ and the ten inequalities $r_i + r_j + s_{ij} \leq t$, and forget the plane. Every configuration is a point of that set, so the set being empty proves $\pi^{\mathrm{nt}}_{5,2} > t$. Summing $(t - r_i - r_j)^2 - s_{ij}^2 \geq 0$ over the ten pairs and substituting the two identities leaves $10t^2 - 8t + 1$ — which is exactly (4) at $n = 5$, $s = 1$, $\sum_k z_k = 0$. So the relaxation is empty precisely when $10t^2 - 8t + 1 < 0$, and its exact ceiling is the larger root, $(4 + \sqrt{6})/10 = 1/c_5 = 0.644949\ldots$: at that t the point $r_i = 1/5$, $s_{ij} = t - 2/5$ satisfies every constraint, and it is the Gram matrix of a regular simplex in $\mathbb{R}^4$, so what the relaxation forgets is planarity. Two consequences. A rational t below that ceiling — for instance $t = 9/14$ — does admit a Positivstellensatz refutation with rational coefficients whose verification is a finite exact computation, but the bound it yields is subsumed by Theorem 7.2, which gives $\pi^{\mathrm{nt}}_{5,2} \geq 1/c_5$ outright and in one line. And no argument in these 15 variables can reach the 2/3 of [1], let alone 5/7; planarity has to be put back, which is the cost estimated in Section 12.

8. Degenerate configurations: an exact answer at every n

Theorem 7.1 gives $2|z_k| \leq \mathrm{pmax}(z)$ for every entry. Call an entry w of z *saturated* when $2|w| = \mathrm{pmax}(z)$, and call z *degenerate* when it has a nonzero saturated entry. The boundary case of that inequality turns out to be completely understood, and it is the first upper bound on act in this paper that holds at every n.

Theorem 8.1 (the degenerate case)**.** *Let z be a zero-sum configuration of size n with a saturated entry $w \neq 0$, and write $M = \mathrm{pmax}(z)$.*

1. Collinearity. *Every entry $x \neq w$ satisfies $|x|\,w + |w|\,x = 0$; that is, every other entry is a non-positive real multiple of w.*
2. The value of δ. $\mathrm{mass}(z) = \mathrm{pmax}(z)$*, so* $\delta(z) = 1$*.*
3. The count. $\mathrm{act}(z) \leq 2n - 3$.
4. Exactness. *For every $n \geq 3$ the maximum of* act *over the non-trivial zero-sum configurations of size n that have a nonzero saturated entry is exactly $2n - 3$, attained by C_n.*
5. Contrapositive. *If $\delta(z) < 1$, or if $\mathrm{act}(z) > 2n - 3$, then every entry satisfies $2|z_k| < \mathrm{pmax}(z)$ strictly.*

Proof sketch. A saturated entry is active with everything, since $\mathrm{pv}(w, x) \geq 2|w| = M$ and $\leq M$. For (1), $\mathrm{pv}(w, x) = 2|w|$ says $|w + x| = |w| - |x|$, an equality in the reverse triangle inequality, and its scale-free form is the stated relation. For (2), summing (1) over the complement and using $\sum_k z_k = 0$ gives $\sum_{k \neq i} |z_k| = |w|$, so $\mathrm{mass}(z) = 2|w| = M$; this is the only place the zero-sum hypothesis is used. For (3), delete w: this peels off exactly the $n - 1$ pairs meeting w, all of them active, by the pair-list form of $e(G) = e(G - v) + \deg(v)$. In what is left every entry is a non-positive multiple of w, so $\mathrm{pv}(x, y) = 2(|x| + |y|)$ there; picking a nonzero entry v of the remainder, a pair avoiding v has $|x| + |y| \leq \mathrm{mass}(z) - |v| < \mathrm{mass}(z)$ and is therefore inactive, so the active pairs of the remainder all meet v and number at most $n - 2$. Then $(n - 1) + (n - 2) = 2n - 3$. (4) adds Theorem 5.1, and (5) is (2) and (3) contraposed. There is no computation anywhere in the argument, and no case analysis on n. ∎

Part (5) is what the next section runs on: either a configuration is degenerate, and Theorem 8.1 answers everything about it, or every entry lies strictly below the saturation threshold, and then the vector model of Section 9 exists. Note that the saturated entry need not be unique — in C_n both 1 and -1 are saturated — but the saturated *value* is: two equal saturated entries w would give $\mathrm{pv}(w, w) = 4|w| = 2M > M$.

9. The vector model, and two structure lemmas

Theorem 9.1 (the vector model). *Let M be real and let $z \in \mathbb{C}$ satisfy $2|z| < M$. Then there is $W \in \mathbb{C} = \mathbb{R}^2$ with*

$$(M - 2|z|)\, W^2 = 2z,$$

and W is unique up to sign; call it a representative *of z at scale M. Let $a, b \in \mathbb{C}$ satisfy $2|a| < M$ and $2|b| < M$, with representatives W_a, W_b. Then:*

(1) *$(M - 2|z|)\,|W|^2 = 2|z|$; in particular $4|z| < M$ if and only if $|W|^2 < 1$;*
(2) *$P_M(a,b)\,\langle W_a, W_b\rangle^2 = q(a,b)$, so $\mathrm{pv}(a,b) \le M \iff \langle W_a, W_b\rangle^2 \le 1$ and $\mathrm{pv}(a,b) = M \iff \langle W_a, W_b\rangle^2 = 1$;*
(3) *if $\langle W_a, W_b\rangle^2 = 1$ then $|W_a|^2|W_b|^2 \ge 1$; hence two entries with $|W| < 1$ are never active with each other, and every active partner of an entry with $|W| < 1$ has $|W| > 1$;*
(4) *if $\mathrm{cross}(W_a, W_b) = 0$ and $\langle W_a, W_b\rangle^2 \le 1$ then $|W_a|^2|W_b|^2 \le 1$; hence among entries with parallel representatives at most one has $|W| > 1$, and two with $|W| = 1$ are active with each other.*

Call an entry z of a configuration *heavy* when $4|z| < \mathrm{pmax}(z)$, i.e. when $|W| < 1$. The whole problem has become an inner-product contact problem for n vectors in the plane: by (2) a pair is active exactly when $\langle W_i, W_j\rangle^2 = 1$, and by (3) and (4) the crude combinatorics of that relation is already restrictive.

Proof sketch. Existence is a square root in the algebraically closed field $\mathbb{C}$, taken of $2z/(M - 2|z|)$; uniqueness up to sign is that W^2 determines W up to sign. (1) is the modulus of the defining equation. (2) is one identity on top of the doubling formula $\langle a^2, b^2\rangle = \langle a, b\rangle^2 - \mathrm{cross}(a,b)^2$ and the Lagrange identity, combined with Theorem 7.1; $P_M > 0$ in this range, so the divisions are legitimate. (3) and (4) are Lagrange read twice: an active pair has $1 = \langle W_a, W_b\rangle^2 \le |W_a|^2|W_b|^2$, and a pair with cross $= 0$ has $|W_a|^2|W_b|^2 = \langle W_a, W_b\rangle^2 \le 1$. Nothing here uses an argument, an arccosine, or a square root of a real number: W is an algebraic square root, and every statement is polynomial. □

Write $\mathrm{adeg}(z, w)$ for the number of positions k of z with $\mathrm{pv}(w, z_k) = \mathrm{pmax}(z)$, the *active degree* of the entry w.

Theorem 9.2 (a heavy entry has active degree at most two). *Let z have length at least 2 and satisfy $2|z_k| < \mathrm{pmax}(z)$ for every entry. If w is an entry with $4|w| < \mathrm{pmax}(z)$, then $\mathrm{adeg}(z, w) \le 2$. There is no convexity, ordering, zero-sum or distinctness hypothesis; by Theorem 8.1(5) the non-degeneracy assumption follows from $\delta(z) < 1$ alone.*

Proof sketch. In the vector model the claim is: if $|W|^2 < 1$ there are no three vectors V_1, V_2, V_3, not necessarily distinct, with $\langle W, V_a\rangle^2 = 1$ for each and $\langle V_a, V_b\rangle^2 \le 1$ pairwise. Put $A = |W|^2 < 1$, $d_a = \langle W, V_a\rangle$, $c_a = \mathrm{cross}(W, V_a)$. The plane Gram identity $A\langle U, V\rangle = \langle W, U\rangle\langle W, V\rangle + \mathrm{cross}(W, U)\,\mathrm{cross}(W, V)$, a polynomial identity in coordinates, gives for $\sigma = d_a d_b$ (so $\sigma^2 = 1$)

$$(d_a c_a)(d_b c_b) = \sigma\, c_a c_b = \sigma\big(A\langle V_a, V_b\rangle - \sigma\big) = \sigma A \langle V_a, V_b\rangle - 1 \le A - 1 < 0,$$

the last step because $(\sigma\langle V_a, V_b\rangle)^2 = \langle V_a, V_b\rangle^2 \le 1$. So the three reals $e_a = d_a c_a$ have all three pairwise products negative, whereas $(e_1e_2)(e_1e_3) = e_1^2(e_2e_3)$ is at once positive and non-positive. Passing back to a configuration needs only that the pairs of a sublist are pairs of the list, applied to the sublist of active partners of w; the entry w never counts itself, because $\mathrm{pv}(w, w) = 4|w| < \mathrm{pmax}(z)$. □

Theorem 9.3 (interior entries are heavy). *Let z satisfy $2|z_k| < M := \mathrm{pmax}(z)$ for every entry, let the pair (z_i, z_j) be active with $\mathrm{cross}(z_i, z_j) \ne 0$, and let an entry z_u lie strictly inside the cone they span,*

$$z_u = \lambda z_i + \mu z_j, \qquad \lambda, \mu > 0,$$

where z_i, z_u, z_j occur at positions of z in this order. Then z_u is heavy: $4|z_u| < M$. (More generally the conclusion holds for any z_i, z_j, z_u with $2|\cdot| < M$ once $\mathrm{pv}(z_i, z_u) \le M$ and $\mathrm{pv}(z_u, z_j) \le M$.)

Proof sketch. In the vector model, with the sign of W_j chosen so that $\langle W_i, W_j\rangle = 1$: Lagrange and $\mathrm{cross}(W_i, W_j) \ne 0$ give $|W_i|^2|W_j|^2 = 1 + \mathrm{cross}(W_i, W_j)^2 > 1$, so one of the two squared norms exceeds 1. If $W_u = \lambda W_i + \mu W_j$ with $\lambda, \mu > 0$, feasibility of the two new pairs reads $\lambda|W_i|^2 + \mu \le 1$ and

$\lambda + \mu|W_j|^2 \le 1$, and whichever norm exceeds 1 turns one of these into $\lambda + \mu < 1$; then $|W_u|^2 = \lambda\langle W_i, W_u\rangle + \mu\langle W_u, W_j\rangle \le \lambda + \mu < 1$. The bridge from the cone in z to the cone in W is the companion of Theorem 9.1(2),

$$2\,\mathrm{cross}(a,b) = (M - 2|a|)(M - 2|b|)\,\langle W_a, W_b\rangle\,\mathrm{cross}(W_a, W_b),$$

together with Cramer's rule: writing $W_u = xW_i + yW_j$, taking $\mathrm{cross}(\cdot, z_i)$ and $\mathrm{cross}(\cdot, z_j)$ of $z_u = \lambda z_i + \mu z_j$ and pushing both through that identity gives $\langle W_i, W_u\rangle\, y > 0$ and $\langle W_u, W_j\rangle\, x > 0$, and a real sign argument whose only input is $|W_i|^2|W_j|^2 > 1$ forces $xy > 0$; replacing W_u by $-W_u$ if necessary puts $x, y > 0$. ☐

Non-parallelism is not decoration: with $W_i = W_j$ a unit vector and $\lambda + \mu = 1$ one gets $|W_u| = 1$ exactly, so the conclusion fails by an equality — and in that case the cone is degenerate and its interior empty. A positive control is the house itself: its edges u_1 and u_3 form an active non-parallel pair, $u_2 = (0, 16) = \frac{2}{5}u_1 + \frac{4}{5}u_3$ lies strictly inside their cone, and indeed $4 \cdot 16 = 64 < 80 = \mathrm{pmax}(H)$; the same edge has active degree exactly 2, so the bound of Theorem 9.2 is attained (Proposition 11.4).

10. An odd-n family

Before this section the counts available were $2n-3$ for C_n at every n (Theorem 5.1), m^2+m-2 for R_m at even $n = 2m$ (Theorem 3.4), and 7 for the house at $n = 5$ (Theorem 3.2). Since C_n ties the house at $n = 5$ and loses at $n = 6$ and $n = 8$ (Proposition 5.3), the natural guess is that $2n - 3$ is the answer at odd n. It is not.

Theorem 10.1 (the family $T(g)$)**.** *Fix $g \ge 2$, let $c \in (0,1)$ be a root of $gc^2 - 3gc + 2 = 0$ — one exists, by the intermediate value theorem, since the quadratic runs from 2 to $2 - 2g < 0$ on $[0,1]$ — and put $s = \sqrt{1-c^2}$ and*

$$v_1 = (c, s), \qquad v_2 = (c, -s), \qquad v_3 = (-2gc, 0),$$
$$T(g) = (\underbrace{v_1, \dots, v_1}_{g}, \underbrace{v_2, \dots, v_2}_{g}, v_3) \quad \textit{of size } n = 2g+1.$$

Then $T(g)$ is an admissible configuration, $|v_1| = |v_2| = 1$, $|v_3| = 2gc \in (\frac{4}{3}, 2)$,

$$\mathrm{pmax}\big(T(g)\big) = 4, \qquad \mathrm{act}\big(T(g)\big) = g^2 + g.$$

Since $g^2 + g > 2n - 3$ for every $g \ge 3$, the configuration C_n is not the odd-n maximiser; at $n = 7$ the maximal number of simultaneously active pairs is at least 12, against $2n - 3 = 11$.

Proof sketch. Only five pair values occur: $\mathrm{pv}(v_1, v_1) = \mathrm{pv}(v_2, v_2) = \mathrm{pv}(v_1, v_3) = \mathrm{pv}(v_2, v_3) = 4$ and $\mathrm{pv}(v_1, v_2) = 2 + 2c < 4$. The one nontrivial value is $\mathrm{pv}(v_1, v_3) = 1 + 2gc + |v_1 + v_3|$ with $|v_1 + v_3|^2 = 1 - 4gc^2 + 4g^2c^2$, and this equals $(3 - 2gc)^2$ exactly when $2gc(3-c) = 4$, which is the defining quadratic in the form $gc(3-c) = 2$; since $3 - 2gc > 1 > 0$ the squaring is reversible. The count is $2\binom{g}{2} + 2g = g^2 + g$, by the same two counting lemmas as Theorem 3.4 applied to the three-block concatenation. The surd c is carried abstractly through its defining quadratic, so one argument covers every g; its values are $(3-\sqrt{5})/2$ at $g = 2$, $(9 - \sqrt{57})/6$ at $g = 3$, $(3 - \sqrt{7})/2$ at $g = 4$. ☐

The construction only wins from $g = 3$: $T(2)$ has $6 < 7 = 2n - 3$ active pairs at $n = 5$, so $n = 5$ stays outside the pattern, consistently with the house's 7.

Theorem 10.2 ($\delta(T(g))$, and odd n)**.** *With the notation of Theorem 10.1,* $\mathrm{mass}(T(g)) = 2g + 2gc$, *so*

$$\delta\big(T(g)\big) = \frac{4}{2g + 2gc} = \frac{2}{g(1+c)}, \qquad \textit{hence} \qquad \pi^{\max}_{2g+1,2} \le \frac{4}{2g+2gc}.$$

Consequently, for every odd $n = 2g + 1 \ge 7$,

$$\pi^{\max}_{n,2} < \frac{4}{n}\cos^2\!\Big(\frac{\pi}{2n}\Big),$$

so the regular n-gon is never optimal at odd $n \geq 7$ either; and at $n = 7$ exactly,

$$\pi^{\max}_{7,2} \leq \frac{4}{15 - \sqrt{57}} = \frac{15 + \sqrt{57}}{42} = 0.5369008\ldots \;<\; \frac{4}{7}\cos^2\!\Big(\frac{\pi}{14}\Big) = 0.5431339\ldots$$

Proof sketch. From $gc(3 - c) = 2$ and $c > 0$ one gets $gc > 2/3$, so $\delta(T(g)) < 6/(3g + 2)$; on the other side $1 - t^2/2 \leq \cos t$ gives $(4/n)\cos^2(\pi/2n) \geq 4/n - \pi^2/n^3$. With $n = 2g + 1$,

$$\frac{4}{n} - \frac{\pi^2}{n^3} - \frac{6}{3g + 2} = \frac{2n^2 - \pi^2(3g + 2)}{n^3(3g + 2)},$$

so everything reduces to $\pi^2(3g+2) < 2(2g+1)^2$, which $\pi < 3.15$ gives for $g \geq 4$. That inequality is *false* at $g = 3$, so $n = 7$ is done separately and exactly: $6c = 9 - \sqrt{57}$ from the quadratic, mass $= 15 - \sqrt{57}$, $\delta = (15 + \sqrt{57})/42$, and $\sqrt{57} < 7.55$ with $\cos(\pi/7) > 0.898$ leave a margin of about 0.0054. ☐

Remark 10.3. $T(g)$ is not claimed to be optimal, and it is not: $\delta(T(g)) \to 4/(n + \frac{1}{3})$ as $g \to \infty$ (because $gc = 2/(3 - c) \to 2/3$; this limit is a one-line consequence of Theorem 10.2 and is not itself part of the formal development), whereas the experiments of [1] suggest $\pi^{\max}_{n,2} \sim 4/(n + \frac{5}{9})$ for odd $n \geq 7$, and $4/(7 + \frac{5}{9}) = 0.529412\ldots$ is below $0.536901\ldots$ So a better odd-n configuration should exist; what Theorem 10.2 supplies is the first explicit one, an exact value at $n = 7$, and a proof of a statement [1] reports only as an experimental suggestion. One coincidence is worth recording: $T(2)$ has $\delta = (5 + \sqrt{5})/10$, *exactly* the regular pentagon's value of Theorem 4.1(3).

11. Controls

Every count and every value above is a statement that a badly stated definition could satisfy vacuously, and every side condition above is one that a reader is entitled to suspect of being decoration. The following checks were computed from the same definitions as the results they guard.

Proposition 11.1 (negative controls for Sections 3–5)**.** (1) Too small, too large. *$\delta(H) \not\leq 0.714$ and $\delta(H) \not\geq 0.715$; $\mathrm{pmax}(H) \not\leq 79$; $\mathrm{act}(H) \notin \{6, 8, 10\}$; $\mathrm{pmax}(R_3) \not\leq 11$ and $\mathrm{pmax}(R_3) \not\geq 13$; $\mathrm{act}(R_3) \notin \{9, 11\}$, out of 15 pairs.*

(2) The hypothesis $m \geq 2$ in Theorems 3.3 and 3.4. *At $m = 1$ the configuration degenerates to $(4i, -4i)$, whose maximum is 8 and not 12, and whose active count is 1 while $m^2 + m - 2 = 0$.*

(3) The hypothesis $x \geq 5$ in Theorem 4.1(1). *At $x = 4$ the inequality is false: $(4/4)\cos^2(\pi/8) = (2 + \sqrt{2})/4 < 6/7 = 4/(4 + \frac{2}{3})$.*

(4) The clauses of admissibility. *The nonzero clause is not vacuous: C_4 is non-trivial but not admissible. The zero-sum clause is not vacuous: $(1, 1)$ fails it. And admissibility is satisfiable with positive mass: H is admissible of size 5 with $\mathrm{mass}(H) = 112 > 0$, so the upper-bound principle used throughout is not applied to an empty hypothesis.*

Item (3) is the sharpest of the four. It is not merely that the proof of Theorem 4.1(1) needs $x \geq 5$; the statement itself fails at $x = 4$, which is why Conjecture 4.4 of [1] begins at $n = 6$, and it means the inequality cannot be extended downward by any argument.

Proposition 11.2 (controls for the maxima at $n \leq 4$)**.** (1) The zero-sum constraint is what forbids six. *For $(1, 1, 1, 1)$ every pair value is $1 + 1 + 2 = 4$, so all six pairs are active, and $\sum_k z_k = 4 \neq 0$. [1] makes the same observation in the same Remark 4.5: "Without the constraint $\sum_k z_k = 0$, all the optimal configurations … all $\binom{n}{d}$ pairs attain the maximum."*

(2) Non-triviality bites. *$(0, 0, 0, 0)$ sums to zero and has all six pairs active (every pair value is $0 = \mathrm{pmax}$); it is exactly the configuration [1, §3.1] excludes, and exactly the one Theorem 6.2 must exclude.*

(3) Neither bound can be lowered. *4 is not an upper bound in the non-trivial class (C_4 reaches 5), and 3 is not one in the admissible class (S reaches 4).*

(4) The too-large claim is refuted. *No non-trivial zero-sum configuration of size 4 has $\mathrm{act} = 6$.*

(5) Theorem 6.1 needs its hypothesis. *For $(1, 1, i)$, which does not sum to zero, $\mathrm{pmax} = 4 \neq 3 = \mathrm{mass}$ and only 1 of the 3 pairs is active.*

Proposition 11.3 (controls for Sections 7 and 10)**.** (1) The side condition of Theorem 7.1 bites. *At* $a = 1$*,* $b = -1$*,* $M = 1$*, where* $2|a| > M$*, one has* $q = 0 \le 1 = P_M$ *while* $\mathrm{pv} = 2 > 1 = M$*.*

(2) The factor 2 in q is not decoration*: dropping it makes* (3) *false at* $a = b = 1$*,* $M = 0$*.* And $2|z_k| \le \mathrm{pmax}$ needs length ≥ 2*: at length* 1*,* $\mathrm{pmax} = 0$*.*

(3) The closure bound is sharp at $n = 3$ and cannot be improved. $c_3 = 1$ *and* $(1, -1, 0)$ *has* mass $= \mathrm{pmax} = 2$*; the same triple refutes* mass $\le (c_n - \frac{1}{10})\,\mathrm{pmax}$*. At length* 1*,* $c_1 = 0$ *while* mass $= 1 > 0$*.*

(4) $T(g)$ needs its quadratic*: at* $g = 3$ *with* $c = 1/2$*, not a root, one gets* $\mathrm{pv}(v_1, v_3) = 4 + \sqrt{7} \ne 4$*.*

(5) $T(g)$ does not activate everything*:* $T(3)$ *activates* 12 *of the* $\binom{7}{2} = 21$ *pairs.* And it does not win at $g = 2$*:* $T(2)$ *has* $6 < 7 = 2n - 3$*.*

(6) The uniform argument of Theorem 10.2 really fails at $g = 3$*:* $\pi^2 \cdot 11 > 2 \cdot 7^2$*, which is why* $n = 7$ *carries its own exact-surd argument.*

(7) The pentagon coincidence is exact*:* $\delta(T(2)) = (5 + \sqrt{5})/10$*.*

Proposition 11.4 (controls for Sections 8 and 9)**.** (1) Theorem 8.1(2) needs the zero-sum hypothesis*: the list* $(1, 0)$ *has a saturated entry and* $\mathrm{pmax} = 2$*, but* $\delta = 2 \ne 1$*.*

(2) Theorem 8.1(3) needs degeneracy*:* $T(3)$ *at* $n = 7$ *has* $12 > 11 = 2n - 3$ *active pairs, so* $2n - 3$ *is not a general bound.* And it cannot be lowered*:* C_n *is degenerate and attains* $2n - 3$ *at every* $n \ge 2$*.*

(3) The saturated entry need not be unique, and the relation is signed*: in* C_4 *both* 1 *and* -1 *are saturated, and each is a* negative *multiple of the other.* And the pair-mass bound is about distinct positions*:* $(1, 1)$ *is not a pair of the one-element list* $[1]$*, where* $|1| + |1| = 2 > 1 =$ mass.

(4) Theorem 9.2 needs heaviness*: in* C_5 *the entry* 1 *is saturated, not heavy, and has active degree* $4 > 2$*.* And the bound 2 is attained*: the house's edge* $(0, 16)$ *has* $4 \cdot 16 = 64 < 80 = \mathrm{pmax}(H)$ *and active degree exactly* 2*.*

(5) Theorem 9.3, positively*: in the house,* $u_2 = (0, 16) = \frac{2}{5}u_1 + \frac{4}{5}u_3$ *with* (u_1, u_3) *active and non-parallel, and* $4 \cdot 16 = 64 < 80$ *as predicted.* Feasibility is load-bearing*:* $W_i = 2$*,* $W_j = \frac{1}{2} + i$ *have* $\langle W_i, W_j\rangle = 1$ *and* cross $= 2 \ne 0$*, but at* $\lambda = \mu = 1$ *one gets* $|W_u|^2 = \frac{29}{4} > 1$ *with* $\langle W_i, W_u\rangle^2 = 25 > 1$*.* And non-parallelism is*: at* $W_i = W_j = 1$*,* $\lambda = \mu = \frac{1}{2}$*, everything holds except* $\mathrm{cross}(W_i, W_j) \ne 0$*, and* $|W_u|^2 = 1$ *exactly.*

12. What is not settled, and why

Optimality. Nothing above touches the optimality halves of Conjectures 4.3 and 4.4 of [1] — that 5/7 is the *minimum* over pentagons and $4/(n + \frac{2}{3})$ over even n. In the pentagon case the missing statement is

$$\forall z \in \mathbb{C}^5 : \quad \sum_k z_k = 0 \ \wedge\ \sum_k |z_k| = 1 \implies \max_{1 \le i < j \le 5} \big(|z_i| + |z_j| + |z_i + z_j|\big) \ \ge\ \tfrac{5}{7}. \tag{5}$$

Its conclusion is a *disjunction* over the ten pairs: for every z, *some* pair is large. It is tempting to conclude from that shape alone that the statement is out of reach of positivity certificates, since no single polynomial is nonnegative here; that conclusion would be wrong, and we record why, because "blocked by shape" and "blocked by cost" are different states and only the second is true.

A disjunction is resolved by branch and bound with a *per-box witness*: on each box of a subdivision it suffices to exhibit one pair (i, j) whose value exceeds the threshold throughout that box, which is an ordinary interval computation, and the boxes carry different witnesses. Equally, negating the min–max turns (5) into an *infeasibility* statement — the system $\sum_k z_k = 0$, $\sum_k |z_k| = 1$, $\mathrm{pv}_{ij} \le c$ for all $i < j$ has no solution — and introducing $r_k \ge 0$ with $r_k^2 = x_k^2 + y_k^2$ and $t_{ij} \ge 0$ with $t_{ij}^2 = |z_i + z_j|^2$ makes that an ordinary polynomial infeasibility problem in 15 variables, for which a single Positivstellensatz certificate is not structurally excluded. What does block both routes is cost and conditioning. Reducing to a box by taking z_1 of largest modulus and rotating and scaling so that $z_1 = 1$, which puts $z_2, z_3, z_4 \in [-1, 1]^2$ with $z_5 = -(1 + z_2 + z_3 + z_4)$ — so that the zero-sum constraint is solved rather than imposed — and running interval arithmetic with widest-coordinate bisection and a per-box witness pair, every threshold tried terminates completely: 9 819 boxes at $c = 0.40$, 113 239 at $c = 0.50$, 2 210 187 at $c = 0.60$, 111 866 363 at $c = 0.68$ and 817 003 135 at $c = 0.70$, the last in 83 seconds of single-threaded C in double precision on an Intel Xeon Silver 4210 at 2.20 GHz. That measurement is a scouting run and not a certificate — a certificate needs directed rounding or exact rational arithmetic, and inside a proof assistant a further factor of 10 to 100 — but it fixes the price: the box count rises by a factor of about 50 from $c = 0.60$ to $c = 0.68$ and by a further factor of about 7 from 0.68 to 0.70, so the entry price for saying anything the literature does not already give (Section 3: the live window starts at 2/3) is 10^8 boxes rather than 10^6. On the certificate route the obstruction is degeneracy and degree: at the house 7 of the 10 constraints are active, so there is no strict complementarity, and adding the planarity that Remark 7.3 shows to be missing means ten degree-6 equalities among the 3×3 minors of the Gram matrix, which pushes the sums-of-squares degree to 6 in 15 variables. Finally, 5/7 itself is out of reach of any pure box method, since the 24 rotated and relabelled images of the house are exact zeros of the margin there; settling Conjecture 4.3 needs branch and bound on the complement of explicit balls around those images together with a certified local analysis inside them. *No optimality computation of any kind was carried out for this paper*; the numbers above are a price, not a result. [1] itself claims only to have "verified that the house is a strict local minimum up to rotation and scaling".

Other open ends.

- *The maximum active count at $n = 5$ and $n = 6$.* Remark 5.5 reports 7 and 10 as the numerical maxima, attained by the house and by R_3; proving either would settle the first half of the question of [1, Remark 4.5] at that n. The $n = 4$ argument of Section 6 does not transfer verbatim, because for $n \geq 5$ the zero-sum relation no longer pairs the indices two by two. We have an unformalised argument for both values, combining Theorem 8.1, Theorems 9.2 and 9.3, a structural upper bound assembled from them, and a finite elimination of the residual structures; it is work in progress, it has not been machine-checked or refereed, and its remaining gap is precisely the bookkeeping over the cyclic order of the positions on the circle $\mathbb{R}/\pi$ that Theorems 9.2 and 9.3 were designed to avoid. We claim nothing for it here.
- *The maximum at general n.* Theorem 8.1 settles the degenerate class at every n and Section 6 settles $n \leq 4$ outright, but no upper bound is proved for non-degenerate configurations of size $n \geq 5$. The data — $g^2 + g$ at odd $n = 2g + 1$ from Theorem 10.1, $m^2 + m - 2$ at even $n = 2m$ from Theorem 3.4, 7 at $n = 5$, $2n - 3$ over the degenerate class — suggest that those two counts are the answers, and Remark 5.5 supports it at $n = 5, 6$; we state that as a question rather than a conjecture, since the only evidence at $n \geq 7$ is a search.
- *Odd n, exactly.* Theorem 10.2 shows the regular n-gon is never optimal at odd $n \geq 7$ and gives the value $(15 + \sqrt{57})/42$ at $n = 7$, but Remark 10.3 shows $T(g)$ is not optimal. Finding the odd-n analogue of the 3-4 rectangle — a configuration approaching $4/(n + \frac{5}{9})$, the value the experiments of [1] suggest — is the natural next target, and it is of exactly the shape of Theorems 3.1 and 3.3.
- *The lower bound $4/(n + 1)$.* Theorem 7.2 is weaker, and the argument of [1, Theorem 3.8] is not reproduced here. Reproducing it is what would put the live window $[2/3, 5/7]$ inside the formally verified part of this paper.

13. Provenance

Theorems 5.2, 6.2, 6.3, 8.1, 9.2, 9.3, 10.1 and 10.2 are the ones offered as new. Theorems 3.1–3.4 certify values and counts that [1] prints; Theorem 4.1 makes exact two assertions that [1] makes without computing; Theorems 3.5 and 7.2 are averaging and summation bounds, both weaker than that paper's own; Theorem 7.1 and Theorem 9.1 are tools — a polynomial identity and a half-angle square root — that nothing in [1] states but that no-one would call new mathematics; and Theorems 5.1 and 6.1 are elementary computations, separated from the results they serve.

Before any of it was formalised, all of the values [1] prints — the perimeter 112, the ratio 5/7, the count 7 of 10, $\delta(R_m) = 4/(n + \frac{2}{3})$ and the count $m^2 + m - 2$ for $m = 2, \ldots, 7$, $\delta(A_4) = (2 + \sqrt{2})/4$, and the 1.3% — were reproduced independently in exact rational arithmetic, and all agree. The one place where the reproduction *disagrees* with a printed assertion is instructive and is recorded as

Proposition 11.1(3): the clause $(4/n)\cos^2(\pi/2n) > 4/(n+2/3)$ is false at $n = 4$, which is consistent with [1] restricting Conjecture 4.4 to $n \geq 6$. Each later group of results was likewise checked numerically before it was proved: an *exhaustive* sweep of the $7^6 = 117\,649$ Gaussian-integer zero-sum configurations with $z_1, z_2, z_3 \in \{-3, \ldots, 3\}^2$, pair values evaluated as 60-digit decimals and compared at tolerance 10^{-40}, which returns maximum 5 over the non-trivial class with all 288 maximisers of shape $(u, -u, 0, 0)$ and maximum 4 once every entry is nonzero (Theorems 6.2 and 6.3); $2 \cdot 10^5$ random (a, b, M) for (3) and $2 \cdot 10^4$ random configurations for the equivalences, with no mismatches; $2 \cdot 10^5$ arbitrary, not necessarily zero-sum, lists for the closure bound, with no violations and supremum ratio exactly 1; $2 \cdot 10^4$ random degenerate configurations for every clause of Theorem 8.1; $4 \cdot 10^4$ random configurations for Theorem 9.2, with active-degree histogram $\{0{:}112042,\ 1{:}25142,\ 2{:}5271\}$; $8\,620$ and $11\,417$ interior cases for the two forms of Theorem 9.3 and $4 \cdot 10^4$ triples confirming that cone membership in the z-plane and in the W-plane agree exactly; and $T(g)$ for $g = 2, \ldots, 8$ both in floating point and symbolically, with $\delta(T(g))$ compared against the regular n-gon for $g = 2, \ldots, 40$ (a tie at $g = 2$, a strict win at every $g \geq 3$). These are gates, not proofs, and none of them appears in an argument above.

For the new results the following was searched, most recently on 2026-09-03. The source has been at version 2 since 2026-08-31; the two e-prints were diffed line by line, and every difference lies in its §1 and §2 or after the bibliography: two resized subfigures with new image files and an added third subfigure inside its Figure 1, one corrected constant in that figure's caption, one added paragraph describing an interactive animation, a two-word deletion in its §1.3, an added unnumbered acknowledgements subsection, and one added full stop in the bibliography file. Its §3 and its §4 are byte-identical in the two versions, so the closing Remark 4.5 is, and no section, result, figure or equation number moves; the question this paper answers pieces of is still posed as open in the current version, and every section and result number cited here is stable across both. The source has no citations recorded on either Semantic Scholar (paper identifier `d87440aa…`) or OpenAlex (`W7203660346`), the latter checked both as a citation count and through a cites filter, which returns nothing. In a local full-text corpus of arXiv preprints with coverage through the 2026-08-29 announcement — 743 shards and about 86 gigabytes, of which 261 shards and 72 gigabytes are mathematics — the phrases "largest triangle perimeter", "maximal triangle perimeter", "two consecutive edges of a convex polygon" and "simultaneously active pairs" occur in no mathematics shard at all, the discriminating string "3-4 rectangle" occurs only inside the source itself, and the identifier 2608.15444 occurs nowhere as a citation. A live arXiv search for "locally positive semidefinite" returns six papers in total, none but the source concerning $\pi^{\max}_{n,2}$, the polygon problem, or active pairs. OEIS has no sequence matching the table $3, 5, 7, 10, 12, 18, 20, 28, 30$ of conjectured maxima or any of its prefixes.

Every section and result number attributed to [1] above was resolved by compiling its e-print and reading the numbers off the compiled document, not by counting sections in the source text: the paper has four numbered sections and two appendices, its §4 is "Conjectures and Open Problems", and the closing open question is its Remark 4.5. That was done for version 1 on 2026-08-30 and re-checked against version 2 on 2026-09-02, where the only addition to the numbered structure is an unnumbered acknowledgements subsection, so no counter moves.

Two warnings bound those negatives. A whole-corpus phrase search is fragile to line wrapping in LaTeX sources — the source's own occurrence of "simultaneously active pairs" is split across a newline in its shard — so a zero there is evidence and not proof; the load-bearing negative is the citation graph together with the fact that the question is three weeks old and posed by the source itself. And the channels *not* searched are the same as before: MathSciNet, zbMATH, Google Scholar, and any publication absent from arXiv and from the corpus. "Not found" is not "new".

Two neighbouring literatures were identified and are recorded so that a later reader does not have to rediscover them. The nearest relative, Nesterenko [2] — which [1] credits with the same complex-squaring reduction — was fetched and read in full: its functional is $\max_{i,j}(|a_i| + |a_j| - |a_i + a_j|)$, with a *minus* sign, the combination occurring in the $d = n - 2$ case of [1] rather than the $d = 2$ functional (1); it contains no pentagon, no value $5/7$ and no active-pair count. A corpus search on that functional turns up a small further line of preprints [3] on the Goreinov–Tyrtyshnikov–Zamarashkin hypothesis on submatrices with best-bounded inverses, whose 2×2 case is the minus-sign twin of the problem here; only the abstracts of those preprints were read, and nothing in this paper depends on them.

Separately, the "small polygons" line [4], which [1] cites as structurally related, was *not* read: its functionals are diameter-constrained perimeter or area rather than the ratio δ, and its optimal pentagons are not lattice polygons. Finally, two coincidences of quadratics: the active counts $4, 10, 18, 28, 40, 54, 70$ of $R_2, \ldots, R_8$ match OEIS A028552 under a shift, and the counts $g^2 + g$ of Theorem 10.1 are the pronic numbers, OEIS A002378 [5]. Neither carries any apparent connection to this problem.

Changes from version 1 of this paper. Nothing stated in version 1 is withdrawn: every theorem, proposition and corollary of that version stands here with its statement unchanged, the only edit to any of them being that Proposition 11.1, called "negative controls" there, is now named for the sections it guards; items (6)–(10) of Section 1 are added. Two things version 1 *said* are corrected, and we label both rather than silently rewriting them. First, version 1 printed the bracket $2/5 \leq \pi_{5,2}^{\max} \leq 5/7$ of Theorem 3.5 without recording that the left end already known is the $4/(n+1)$ of [1, Theorem 3.8]: the live window is $2/3 \leq \pi_{5,2}^{\max} \leq 5/7$, and it is that window, not the printed one, that Conjecture 4.3 of [1] has to close. The abstract, item (4) of Section 1 and Section 3 now say so. Second, version 1 asserted that the disjunction shape of (5) is itself what puts the optimality direction out of reach of positivity certificates. That is wrong, as Section 12 now sets out: a disjunction is resolved by a per-box witness, and the negation is an ordinary polynomial infeasibility problem; the obstruction is cost and conditioning. Finally, the count of Lean declarations in Section 14 is stated under a stricter convention than version 1's, which is said there.

14. Verification

Every theorem, proposition and corollary above has been formally verified in Lean 4 with Mathlib, in the directory `LeanProblemSpec/ActivePairs/` of a repository that is private at the time of writing (repository reference to be added at submission); the development is fourteen modules, in which 128 declarations carry the marker that binds a Lean statement to one of the numbered results above, and it contains no `sorry` and no declared axiom. We count markers, and say so because version 1 of this paper did not: it reported 46 for its six modules, which is the number of declarations there whose axioms were listed one by one, against 40 markers over the same six modules. The exceptions are the record collected in Remark 5.5, the certificate discussed in Remark 7.3, the unformalised argument mentioned in Section 12, and the one-line observations of Remarks 5.4 and 10.3 and of Section 12, each of which says at the point where it stands that it is outside the formal development. Three features of that development are worth naming because they are unusual. *No theorem or proposition here rests on an exhaustive computation*: there is no enumeration, no search, and no appeal to compiled evaluation — Lean's `native_decide` is not used anywhere, and `#print axioms` returned exactly the axiom set {`propext`, `Classical.choice`, `Quot.sound`} on every one of the 150 declarations it was run on, the marked statements together with the lemmas they rest on, with `Classical.choice` entering only through the real numbers (their order and the infimum (2)) and through the classical decidability of equality inside the active-pair count. The certificate of Remark 7.3 is the one place where a finite exact computation is checked by the kernel, and it stands outside the numbered results for that reason among others. And *no floating-point arithmetic enters at any point*: the two configurations of [1] are lattice configurations whose side and diagonal lengths are integers, so every value of pv in Sections 3 and 5 is an integer certified by three Pythagorean identities over $\mathbb{Z}$, and each irrational quantity that occurs — $\sqrt{241}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{57}$ and π — is handled by one explicit rational bound ($\sqrt{241} < 39$; $\sqrt{2} < 1.415$; $2.236 < \sqrt{5} < 2.2361$; $\sqrt{57} < 7.55$; $\pi < 3.15$) or by an exact identity ($\cos(\pi/5) = (1 + \sqrt{5})/4$); the quadratic surd c of Theorem 10.1 is never evaluated at all, but carried abstractly through its defining equation, which is what lets one argument cover every g. Counting arguments that a reader might expect to be done n by n — the m^2+m-2 of Theorem 3.4, the $2n-3$ of Theorem 5.1, the g^2+g of Theorem 10.1 and the $2n-3$ of Theorem 8.1 — are proved uniformly, from three general lemmas about counting over the unordered pairs of a list, the third of which is the pair-list form of $e(G) = e(G-v) + \deg(v)$. The finite searches that were run are all outside the formal development and all reported as such in Section 13 and in Remark 5.5, and their negatives are not proofs. The cost of the formal part, measured module by module in import order with one check each on a 20-core Intel Xeon Silver 4210 at 2.20 GHz shared with other jobs, is 11.6, 10.6, 10.2, 60.2, 177.5, 12.9, 33.8, 10.3, 23.7, 10.7, 11.7, 16.6, 17.3 and 69.4 seconds of wall clock — about eight minutes in total — with peak resident sets between 6.78 and 7.08 GB against a baseline of about 6.9 GB for importing the library alone, so the marginal memory cost is essentially zero. Those are wall-clock figures on a loaded machine and are therefore upper bounds: the seventh, the $n = 4$ module, was re-checked once for this version and spent 21.2 seconds of processor time inside its 33.8 seconds of wall clock. All of it builds under the Lean toolchain `leanprover/lean4:v4.32.0` with Mathlib pinned at its `v4.32.0` tag.

References

[1] J. Acevedo, G. Blekherman, S. Debus, S. Lee and C. Riener, *Eigenvalues of locally positive semidefinite matrices: Non-convexity and Geometry*, arXiv:2608.15444 (version 1, 15 August 2026; version 2, 31 August 2026). Primary arXiv category `math.RA`, cross-listed `math.AG` and `math.OC`; MSC 2020: primary 15A99, secondary 14P10, 90C47. All quotations and all result numbers here were first read from the version 1 LaTeX e-print and its compiled form on 2026-08-30 — §4 in full, as were §3.1 and Theorems 3.8, 3.12, 3.16 and 3.17, which are cited but not reproved, and Figures 3a and 3b, which were read from the e-print's vector graphics — and re-checked against the version 2 e-print on 2026-09-02 and 2026-09-03. The two versions agree byte for byte on §3 and §4, so Remark 4.5 is byte-identical and no numbering moves; the changes are the ones listed in Section 13, all in §1–§2 or after the bibliography. One of them is the authors' own acknowledgement, added in version 2, of AI assistance in formulating parts of the proofs of their Proposition 2.1, Theorem 3.8 and Theorem 3.16, with the final proofs written by them; the last two are cited but not reproved here, Theorem 3.16 being what Theorem 5.2 rests on. No journal reference is recorded on its arXiv abstract page, checked 2026-09-03.

[2] Yu. Nesterenko, *Submatrices with the best-bounded inverses: Studying* $\mathbb{R}^{n\times 2}$ *and* $\mathbb{C}^{n\times 2}$, arXiv:2408.16631 (2024). Cited by [1] for the same complex-squaring reduction; read in full here. Its functional is $\max_{i,j}(|a_i| + |a_j| - |a_i + a_j|)$, not the functional of this paper. Preprint only: no journal version is recorded on its arXiv abstract page, in Crossref or in dblp, all checked 2026-08-30.

[3] The preprint line on the Goreinov–Tyrtyshnikov–Zamarashkin hypothesis on submatrices with best-bounded inverses — whose 2×2 case is the minus-sign twin of the functional (1) — located by a full-text corpus search on 2026-09-03 alongside [2]: Yu. Nesterenko, *Submatrices with the best-bounded inverses: revisiting the hypothesis*, arXiv:2303.07492 (v3, 2024); R. Sengupta and M. Pautov, *On the submatrices with the best-bounded inverses*, arXiv:2604.05944 (v5, 2026); Yu. Nesterenko, *Submatrices with the best-bounded inverses: the equality criterion for* $\mathbb{R}^{n\times 2}$, arXiv:2604.14050 (v2, 2026); Yu. Nesterenko, *Submatrices with the best-bounded inverses: an asymptotically tight upper bound for* $\mathbb{C}^{n\times 2}$, arXiv:2604.24087 (2026). Titles, authors and versions were read from the arXiv API on 2026-09-03; all four are `math.NA` preprints, and no journal version is recorded for any of them on its arXiv abstract page, in Crossref or in OpenAlex, all checked the same day. Only their abstracts were read here; nothing above depends on them.

[4] D. Henrion and F. Messine, *Finding largest small polygons with GloptiPoly*, J. Global Optim. 56 (2013), no. 3, 1017–1028; C. Bingane and M. J. Mossinghoff, *Small polygons with large area*, J. Global Optim. 88 (2024), no. 4, 1035–1050; B. Mulansky and A. Potschka, *A zonogon approach for computing small convex polygons of maximum perimeter*, Math. Program. 218 (2026), no. 1–2, 563–589, with a correction, ibid., 593–596. Cited from [1] as the structurally related "small polygons" line; not read here. Their functionals are diameter-constrained perimeter or area rather than the ratio δ of (1), and their optimal pentagons are not lattice polygons. Volumes, issues, pages and the year of the printed issue are from the publishers' Crossref records (`doi:10.1007/s10898-011-9818-7`, `doi:10.1007/s10898-023-01329-1`, `doi:10.1007/s10107-025-02244-x` and `doi:10.1007/s10107-025-02257-6`), corroborated for the last two by dblp, and not from the bibliography of [1], which cites the third under its preprint title and as "online first".

[5] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`, entries A028552 ($n(n+3)$), read 2026-08-30, and A002378 ($n(n + 1)$, the pronic numbers), read 2026-09-03. Cited in Section 13 only to record coincidences of quadratics, not connections.

[6] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381, `doi:10.1145/3372885.3373824`. The version used here is the `v4.32.0` tag.