ACTIVE PAIRS OF A MIN–MAX PROBLEM ON ZERO-SUM PLANAR CONFIGURATIONS: THE HOUSE PENTAGON, THE 3-4 RECTANGLE, AND AN OPTIMUM THAT IS NOT ACTIVE-MAXIMAL

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ABSTRACT. For $z_1, \dots, z_n \in \mathbb{C}$ summing to zero put $\mathrm{pv}(z_i, z_j) = |z_i| + |z_j| + |z_i + z_j|$ and $\delta(z) = \max_{i<j} \mathrm{pv}(z_i, z_j) / \sum_k |z_k|$; the quantity $\pi^{\max}_{n,2} = \inf_z \delta(z)$, the infimum taken over the z with every $z_k \neq 0$, measures where the eigenvalue set of the locally positive semidefinite matrices fails to be convex, and in the polygon dictionary of Acevedo, Blekherman, Debus, Lee and Riener it asks how small the largest two-edge triangle perimeter of a convex n-gon can be relative to its perimeter. A pair (i, j) is *active* when it attains the maximum. We certify, in exact integer arithmetic, the two configurations those authors conjecture to be optimal: the *house* pentagon has perimeter 112 and largest two-edge triangle perimeter 80, so $\delta = 5/7$ exactly and $\pi^{\max}_{5,2} \le 5/7$, with 7 of its 10 pairs active; the 3-4 *rectangle* at $n = 2m$, $m \ge 2$, has perimeter $6m + 2$ and maximum 12, so $\delta = 4/(n + \frac{2}{3})$ exactly and $\pi^{\max}_{n,2} \le 4/(n + \frac{2}{3})$, with $m^2 + m - 2$ of its $2m^2 - m$ pairs active — the last uniformly in m, not one n at a time. We prove the comparison $4/(x + \frac{2}{3}) < (4/x)\cos^2(\pi/2x)$ for every real $x \ge 5$ that those authors assert in one clause, so that the regular n-gon is never optimal for even $n \ge 6$, and we pin their "approximately 1.3%" at the pentagon to the exact algebraic number $(5\sqrt{5} - 11)/14 \in (0.0128, 0.0129)$. A uniform bound $\delta(z) \ge 2/n$ makes the brackets $2/5 \le \pi^{\max}_{5,2} \le 5/7$ and $1/m \le \pi^{\max}_{2m,2} \le 4/(2m + \frac{2}{3})$ $(m \ge 2)$ non-empty. Our main new result answers the second half of their closing open question at $n = 4$, in the negative: the square, which they prove to be the unique optimal configuration there, activates 4 pairs, while the non-trivial zero-sum configuration $(1, -1, 0, 0)$ activates 5. The phenomenon is confined to $n = 4$. Every theorem below is machine-checked in Lean 4; the development uses no floating-point arithmetic and no compiled evaluation.

1. Introduction

The objects. A *configuration* of size n is a tuple $z = (z_1, \dots, z_n)$ of complex numbers with $\sum_k z_k = 0$. Write

$$\mathrm{pv}(a, b) = |a| + |b| + |a + b|, \qquad \mathrm{mass}(z) = \sum_{k=1}^{n} |z_k|, \qquad \mathrm{pmax}(z) = \max_{1 \le i < j \le n} \mathrm{pv}(z_i, z_j), \tag{1}$$

and $\delta(z) = \mathrm{pmax}(z) / \mathrm{mass}(z)$. Both pmax and mass are homogeneous of degree one and invariant under a global rotation, so δ is a scale- and rotation-invariant functional on configurations, and the min–max problem

$$\pi^{\max}_{n,2} = \inf\{\delta(z) : \ z \text{ a configuration of size } n \text{ with every } z_k \neq 0\} \tag{2}$$

is a problem about shapes. A pair (i, j) with $i < j$ is called *active* at z when $\mathrm{pv}(z_i, z_j) = \mathrm{pmax}(z)$, and we write $\mathrm{act}(z)$ for the number of active pairs.

The quantity (2) is due to Acevedo, Blekherman, Debus, Lee and Riener [1], who arrive at it from a spectral question. Let $\lambda(S^{n,2})$ be the set of eigenvalue vectors of the real symmetric $n \times n$ matrices all of whose 2×2 principal submatrices are positive semidefinite. That set is not convex, and [1] shows that $\pi^{\max}_{n,2}$ measures where it fails to be (their Problem 3.1, Proposition 3.3 and Remark 3.4), reaching (2) through a "complex squaring" reduction that also appears, for a different functional, in Nesterenko [2].

What makes (2) a geometry problem is the dictionary of [1, §4]. Sorted by argument, the numbers $z_1, \dots, z_n$ of a configuration are the edge vectors of a closed convex polygonal path $A_1 A_2 \cdots A_n$; then $\mathrm{mass}(z)$ is the perimeter of that polygon, and for two edge vectors z_i, z_j the number $\mathrm{pv}(z_i, z_j)$ is the perimeter of the triangle with sides z_i, z_j and $-(z_i + z_j)$ — for consecutive indices, the triangle

$A_kA_{k+1}A_{k+2}$ cut off by the diagonal A_kA_{k+2}. So $\pi_{n,2}^{\max}$ asks: *over all convex n-gons, how small can the largest two-edge triangle perimeter be, relative to the perimeter of the polygon?*

What [1] **conjectures, and what it leaves open.** Section 4 of [1] is a list of conjectures and open problems, and three of its items are the starting point here. For $n = 5$:

> "*The house*, corresponding to a convex pentagon $ABCDE$ where $ABCE$ is a rectangle and $|AB| = 30$, $|BC| = |EA| = 16$, $|CD| = |DE| = 25$ (see Figure 3a), is an optimal polygon with $\pi_{5,2}^{\max} = 5/7$. We have verified that the house is a strict local minimum up to rotation and scaling." [1], Conjecture 4.3

For even n, the 3-4 *rectangle*: take $(n-2)/2$ of the z_k equal to 3, as many equal to -3, one equal to $4i$ and one equal to $-4i$; then $\pi_{n,2}^{\max} = 4/(n + \frac{2}{3})$ for even $n \geq 6$ [1, Conjecture 4.4]. Both conjectures have two halves. The *value* half — that the stated configuration has the stated δ — is an identity about one explicit object. The *optimality* half — that no configuration does better — is a statement about all of $\mathbb{C}^n$. Since $\pi_{n,2}^{\max}$ is an infimum, the value half alone already gives an unconditional *upper* bound on $\pi_{n,2}^{\max}$; the reverse inequality is the conjecture.

The third item is [1, Remark 4.5], which closes its §4. Having observed that both conjecturally optimal configurations activate roughly half of all pairs — the house 7 of $\binom{5}{2} = 10$, the 3-4 rectangle $m^2 + m - 2$ of $2m^2 - m$ where $n = 2m$ — the authors write:

> "Determining the maximal number of simultaneously active pairs for a non-trivial configuration with $\sum_k z_k = 0$, and whether the optimal configurations for $\pi_{n,2}^{\max}$ always achieve this maximum is an interesting open question." [1], Remark 4.5

"Non-trivial" is their own term, fixed earlier in the paper: "the conjecture always holds for the trivial configuration $z_1 = \cdots = z_n = 0$, so for the remainder of this paper, we will implicitly exclude this configuration" [1, §3.1]. Only the all-zero configuration is excluded, and individual z_k are allowed to vanish — as they are in the class over which [1, Proposition 3.3] takes the minimum defining its own version of (2). Section 2 states both conventions and compares them; that class is what the present paper's main result turns on.

What this paper settles.

(1) **The house, exactly** (Theorems 3.1 and 3.2). With the lattice coordinates $A = (0, 0)$, $B = (30, 0)$, $C = (30, 16)$, $D = (15, 36)$, $E = (0, 16)$, the five edge vectors sum to zero, the perimeter is 112, the largest two-edge triangle perimeter is 80, so $\delta = 5/7$ *exactly* and $\pi_{5,2}^{\max} \leq 5/7$; the pentagon is strictly convex, and exactly 7 of the 10 pairs are active, the three others having values $41 + \sqrt{241}$ (twice) and 32.

(2) **The** 3-4 **rectangle, exactly and uniformly in** m (Theorems 3.3 and 3.4). At $n = 2m$ with $m \geq 2$ the perimeter is $6m+2$, the maximum is 12, so $\delta = 4/(n+\frac{2}{3})$ exactly and $\pi_{n,2}^{\max} \leq 4/(n+\frac{2}{3})$ for *every* even $n \geq 4$; and exactly $m^2 + m - 2$ of the $2m^2 - m$ pairs are active. Both are proved for general m in one argument, not one n at a time.

(3) **The regular polygon comparison** (Theorem 4.1). The inequality

$$\frac{4}{x + \frac{2}{3}} < \frac{4}{x} \cos^2\!\left(\frac{\pi}{2x}\right),$$

which [1] asserts in one clause without computing it, holds for every *real* $x \geq 5$, so the regular n-gon is never optimal for even $n \geq 6$. At the pentagon, $(4/5)\cos^2(\pi/10) = (5+\sqrt{5})/10$ exactly, and the house improves on it by $(5\sqrt{5} - 11)/14$, which pins the "approximately 1.3%" of [1] to the interval $(0.0128, 0.0129)$.

(4) **A bracket** (Theorem 3.5). Every configuration of $n \geq 2$ nonzero vectors summing to zero has $\delta(z) \geq 2/n$, so the upper bounds above are not vacuous: $2/5 \leq \pi_{5,2}^{\max} \leq 5/7$ and $1/m \leq \pi_{2m,2}^{\max} \leq 4/(2m + \frac{2}{3})$.

(5) **The answer at** $n = 4$ (Theorem 5.2). This is the new mathematics. The square is, by [1, Theorem 3.16] at $n = 4$, the unique optimal configuration for $\pi_{4,2}^{\max} = (2+\sqrt{2})/4$, and it activates 4 of its 6 pairs; the non-trivial zero-sum configuration $(1, -1, 0, 0)$ activates 5. So at $n = 4$ an

optimal configuration does *not* achieve the maximal number of simultaneously active pairs, and the second half of the question quoted above has a negative answer there. The phenomenon is confined to $n = 4$: the same degenerate family ties the house at $n = 5$ and loses at $n = 6$ and $n = 8$ (Proposition 5.3).

Item (5) is the one offered as new; items (1)–(4) reproduce or make exact what [1] prints, and Section 8 says which is which and what search was run. Nothing here touches the optimality halves of Conjectures 4.3 and 4.4; Section 7 says precisely why not.

2. The problem in coordinates

Definition 2.1 (configurations)**.** Fix $n \geq 1$. A tuple $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is

- *non-trivial* when $\sum_k z_k = 0$ and $z \neq 0$ — the term of [1, §3.1];
- *admissible* when $\sum_k z_k = 0$ and $z_k \neq 0$ for every k.

Admissible is strictly stronger, and the two differ exactly on the configurations with some vanishing entry. Throughout, $\pi^{\max}_{n,2}$ is the infimum (2) of δ over the *admissible* configurations of size n.

Which class [1] uses. Definition 2.1 is stricter than the source's own convention, and we say so explicitly because the gap between the two classes is the subject of Section 5. [1, Proposition 3.3] defines, for $2 \leq d \leq n$,

$$\pi^{\max}_{n,d} := \min\Big\{ \max_{S \in \binom{[n]}{d}} \Big(\sum_{s \in S} |z_s| + \Big| \sum_{s \in S} z_s \Big| \Big) \ :\ z \in \mathbb{C}^n,\ \sum_k |z_k| = 1,\ \sum_k z_k = 0 \Big\},$$

and its feasible set — written $\mathcal{Z}_n$ in the proof of [1, Theorem 3.16] — puts *no* condition on the individual z_k: the normalisation $\sum_k |z_k| = 1$ excludes the all-zero tuple and nothing else. At $d = 2$ the objective is $\mathrm{pmax}(z)$ and the normalisation makes $\mathrm{mass}(z) = 1$, so the source's quantity is the infimum of δ over the *non-trivial* configurations. The source calls it $\pi^{\max}_{n,2}$, which is also our symbol for (2); for this one comparison write $\pi^{\mathrm{nt}}_{n,2}$ for the source's. It runs over the larger class, so

$$\pi^{\max}_{n,2} \ \geq \ \pi^{\mathrm{nt}}_{n,2},$$

with equality at every n at which some *admissible* configuration attains $\pi^{\mathrm{nt}}_{n,2}$. At $n = 4$ the square does, so $\pi^{\max}_{4,2} = \pi^{\mathrm{nt}}_{4,2} = (2 + \sqrt{2})/4$ (Theorem 5.1 with [1, Corollary 2.7, Proposition 3.3]). That equality at $n = 4$ is the only identification of the two made anywhere below, and it is what lets Section 5 write $\pi^{\max}_{4,2}$ for the quantity whose optimum [1] classifies. At every other n, $\pi^{\max}_{n,2}$ carries only bounds that are certified by an admissible configuration, and so hold of $\pi^{\mathrm{nt}}_{n,2}$ as well, or that are stated for $\pi^{\max}_{n,2}$ alone (Theorem 3.5). Section 4 of [1] restates the same quantity in polygon language, as $\min\{\delta(\mathcal{X}) : \mathcal{X}$ a "non-trivial convex n-gon"$\}$ with δ introduced "for a nondegenerate polygon" [1, Problem 4.1]; neither quoted term is defined there, and in polygon language a vanishing edge vector is hard to admit, so that phrasing pulls the other way. We take the displayed formula, which is unambiguous, to be the source's convention.

Two remarks on Definition 2.1 keep the statements below aligned with [1]. First, admissibility does *not* ask that consecutive edge vectors be non-parallel. It cannot: the 3-4 rectangle of [1, Conjecture 4.4] has $2m$ edge vectors taking only four distinct values, so as a polygon it is a $3(m-1) \times 4$ rectangle whose two long sides are each subdivided into $m - 1$ edges of length 3. Requiring non-parallel consecutive edges would exclude that paper's own conjectured optimum. Second, every configuration exhibited below is admissible, hence non-trivial; since both infima run over classes containing it, each of them certifies the *same* upper bound for $\pi^{\max}_{n,2}$ and for $\pi^{\max}_{n,2}$, and the upper bounds of Sections 3 and 4 are therefore insensitive to which class one takes. The distinction matters in exactly one place, Section 5, and there it is the point.

The two configurations of [1] are *lattice* configurations: their entries are Gaussian integers, and — this is the arithmetic accident the whole exact treatment runs on — every $|z_k|$ and almost every $|z_i + z_j|$ occurring in them is an integer. We write $\mathrm{pt}(a, b) = a + bi$ and record the evaluation used throughout: if $a^2 + b^2 = c^2$ with $c \geq 0$ then $|\mathrm{pt}(a, b)| = c$, so that a value of pv at two lattice points is determined by three Pythagorean identities over $\mathbb{Z}$.

Finally, for orientation, put $\operatorname{cross}(a,b) = \operatorname{Re}(a)\operatorname{Im}(b) - \operatorname{Im}(a)\operatorname{Re}(b)$, twice the signed area of the triangle spanned by a and b. All consecutive $\operatorname{cross}(z_k, z_{k+1})$ being positive says that the closed path turns strictly left at every vertex, i.e. that it is a strictly convex polygon traversed counterclockwise.

3. The two conjecturally optimal configurations, exactly

3.1. The house.

Theorem 3.1 (the house). *Let $H = (u_1, \dots, u_5)$ with*

$$u_1 = 30, \quad u_2 = 16i, \quad u_3 = -15 + 20i, \quad u_4 = -15 - 20i, \quad u_5 = -16i,$$

the edge vectors of the pentagon $A = (0,0)$, $B = (30,0)$, $C = (30,16)$, $D = (15,36)$, $E = (0,16)$. Then H is an admissible configuration of size 5; its side lengths are $30, 16, 25, 25, 16$, so $\operatorname{mass}(H) = 112$; and $\operatorname{pmax}(H) = 80$. Consequently

$$\delta(H) = \frac{80}{112} = \frac{5}{7} \quad \textit{exactly}, \qquad \textit{and} \qquad \pi^{\max}_{5,2} \le \frac{5}{7}.$$

Moreover H is strictly convex and counterclockwise: $\operatorname{cross}(u_1,u_2) = 480$, $\operatorname{cross}(u_2,u_3) = 240$, $\operatorname{cross}(u_3,u_4) = 600$, $\operatorname{cross}(u_4,u_5) = 240$ and $\operatorname{cross}(u_5,u_1) = 480$.

The value 5/7 is [1, Conjecture 4.3]; what is proved here is the identity, and with it the unconditional upper bound. The coordinates are not printed in [1], which describes the house verbally — "$ABCE$ is a rectangle and $|AB| = 30$, $|BC| = |EA| = 16$, $|CD| = |DE| = 25$" — and otherwise only draws it. The description already fixes the shape up to similarity: the rectangle $ABCE$ is 30×16, and the roof CDE has base $CE = 30$ and equal sides 25, so its height is $\sqrt{625 - 225} = 20$, the 15-20-25 triangle. What it does not fix is the placement, and for that we read the source's Figure 3a — not from the printed page but from the vector graphics of the version 1 e-print. The stroked path of its `house.pdf` has the five vertices, in the drawn cyclic order and in PostScript points,

$$(-85.039, -102.047),\ (85.039, -102.047),\ (85.039, -11.339),\ (0, 102.047),\ (-85.039, -11.339),$$

which at 5.6693 points per unit is exactly A, B, C, D, E as above: apex over the midpoint of AB, traversed counterclockwise, neither reflected nor otherwise placed. The companion Figure 3b draws $u_1, \dots, u_5$ from a common origin, and on the same scale its five arrows are $(30,0)$, $(0,16)$, $(-15,20)$, $(-15,-20)$, $(0,-16)$ — the tuple H itself.

Proof sketch. Each of the five side lengths is a Pythagorean evaluation over $\mathbb{Z}$ ($30^2 + 0^2 = 30^2$, $0^2 + 16^2 = 16^2$, $15^2 + 20^2 = 25^2$ twice, $0^2 + 16^2 = 16^2$), and their sum is 112. For pmax, all ten values of pv are computed in Theorem 3.2; seven of them are 80 and the other three are smaller, the comparison resting on the single irrationality $\sqrt{241} < 39$. Convexity is five integer determinants. All of this is integer arithmetic on ten integers; no search of any kind is involved. □

Theorem 3.2 (the ten pair values). *In lexicographic order of the index pair, the ten values $\operatorname{pv}(u_i, u_j)$ of the house are*

$$80,\ 80,\ 80,\ 80,\ 80,\ 41 + \sqrt{241},\ 32,\ 80,\ 41 + \sqrt{241},\ 80.$$

Exactly seven of them equal $\operatorname{pmax}(H) = 80$, so $\operatorname{act}(H) = 7$ out of $\binom{5}{2} = 10$ pairs. The three inactive pairs are $\{u_2, u_4\}$ and $\{u_3, u_5\}$, of value $41 + \sqrt{241} \approx 56.52$, and $\{u_2, u_5\}$, of value 32.

The count 7 of 10 is the one [1, Remark 4.5] prints; the full list is more than the count, and it isolates what the count depends on. Table 1 records the arithmetic. Thirteen of the fifteen moduli involved — five side lengths and ten pairwise sums — are integers; the exceptions are the two sums $u_2 + u_4 = -15 - 4i$ and $u_3 + u_5 = -15 + 4i$, both of modulus $\sqrt{241}$. The whole separation of active from inactive pairs is the inequality $\sqrt{241} < 39$, i.e. $241 < 1521$.

pair	$\lvert u_i\rvert + \lvert u_j\rvert + \lvert u_i+u_j\rvert$	value	pair	$\lvert u_i\rvert + \lvert u_j\rvert + \lvert u_i+u_j\rvert$	value
$\{1,2\}$	$30+16+34$	80	$\{2,4\}$	$16+25+\sqrt{241}$	≈ 56.52
$\{1,3\}$	$30+25+25$	80	$\{2,5\}$	$16+16+0$	32
$\{1,4\}$	$30+25+25$	80	$\{3,4\}$	$25+25+30$	80
$\{1,5\}$	$30+16+34$	80	$\{3,5\}$	$25+16+\sqrt{241}$	≈ 56.52
$\{2,3\}$	$16+25+39$	80	$\{4,5\}$	$25+16+39$	80

TABLE 1. The ten two-edge triangle perimeters of the house. Seven equal 80; the polygon perimeter is 112, so $\delta = 5/7$.

3.2. **The 3-4 rectangle.**

Theorem 3.3 (the 3-4 rectangle)**.** *For $m \geq 1$ let R_m be the configuration of size $2m$ consisting of $m-1$ entries equal to 3, then one entry $4i$, then $m-1$ entries equal to -3, then one entry $-4i$. Then R_m is admissible,* $\mathrm{mass}(R_m) = 6m+2$, *and* $\mathrm{pmax}(R_m) = 12$ *for every $m \geq 2$. Consequently, writing $n = 2m$,*

$$\delta(R_m) = \frac{12}{6m+2} = \frac{4}{n+\frac{2}{3}} \quad \textit{exactly}, \qquad \textit{and} \qquad \pi^{\max}_{n,2} \leq \frac{4}{n+\frac{2}{3}} \quad \textit{for every even } n \geq 4.$$

Three things are worth separating out. The value $4/(n+\frac{2}{3})$ is [1, Conjecture 4.4], stated there for even $n \geq 6$; the bound above also holds at $n = 4$, where it is *not* sharp, since $\pi^{\max}_{4,2} = (2+\sqrt{2})/4 \approx 0.85355$ is below $4/(4+\frac{2}{3}) = 6/7 \approx 0.85714$ (Section 2). Admissibility is part of the statement and is not automatic: the $2m$ entries take only four distinct values, so as a polygon R_m is a degenerate $2m$-gon (Section 2). And the bound is proved once for all m, not m by m.

Proof sketch. The whole content is the 3-4-5 triangle: $|3|+|4i|+|3+4i| = 3+4+5 = 12$, which is also $|3|+|3|+|3+3| = 12$. Only four values occur among the entries, so only ten unordered value pairs can occur at all, of which the eight realisable at *distinct* positions are

$$(3,3)\colon 12, \quad (3,4i)\colon 12, \quad (3,-3)\colon 6, \quad (3,-4i)\colon 12,$$
$$(-3,-3)\colon 12, \quad (4i,-3)\colon 12, \quad (4i,-4i)\colon 8, \quad (-3,-4i)\colon 12.$$

Each is again a Pythagorean evaluation over $\mathbb{Z}$. The one delicate point is that $(4i,4i)$, worth $16 > 12$, does not occur — and it fails to occur only because $4i$ appears once; the argument is a pairwise statement over the concatenation $3^{m-1}\,4i\,(-3)^{m-1}\,(-4i)$ that keeps this visible rather than hiding it in a case count. Attainment is the pair $(3,4i)$, present as soon as $m \geq 2$. The mass is $3(m-1)+4+3(m-1)+4 = 6m+2$. □

Theorem 3.4 (the active count of the 3-4 rectangle)**.** *For every $m \geq 1$ the configuration R_m has $\binom{2m}{2} = 2m^2 - m$ pairs, and for every $m \geq 2$ exactly m^2+m-2 of them are active. The inactive pairs are the $(m-1)^2$ pairs $\{3,-3\}$, of value 6, together with the single pair $\{4i,-4i\}$, of value 8.*

Proof sketch. The count is $2\binom{m-1}{2} + 4(m-1) = (m-1)(m+2) = m^2+m-2$, and it is obtained uniformly in m from two elementary lemmas about counting over unordered pairs of a list, proved here for this purpose: the number of pairs satisfying a predicate splits over a concatenation $A +\!\!+\, B$ into the pairs inside A, the pairs inside B, and the $|A|\,|B|$ crossing pairs; and a constant block of length n contributes $\binom{n}{2}$ or 0 according to whether the predicate holds of that constant pair. Equivalently, the complement is counted: $2m^2 - m - (m-1)^2 - 1 = m^2+m-2$. No value of m is enumerated. □

3.3. **A bracket.** Every statement so far is an upper bound on an infimum, which says nothing at all unless the infimum is known to be positive. The following makes the brackets non-empty without importing the sharper lower bound of [1].

Theorem 3.5 (a uniform lower bound)**.** *Let z be an admissible configuration of size $n \geq 2$. Then $\delta(z) \geq 2/n$. Consequently $2/n \leq \pi^{\max}_{n,2}$ whenever an admissible configuration of size n exists, and in* *particular*

$$\frac{2}{5} \le \pi^{\max}_{5,2} \le \frac{5}{7}, \qquad \frac{1}{m} \le \pi^{\max}_{2m,2} \le \frac{4}{2m+\frac{2}{3}} \quad (m \ge 2).$$

Proof sketch. From $|a+b| \ge 0$ we get $\mathrm{pv}(a,b) \ge |a|+|b|$. Summing over all $\binom{n}{2}$ pairs, each index appears in $n-1$ of them, so $\sum_{i<j} \mathrm{pv}(z_i,z_j) \ge (n-1)\,\mathrm{mass}(z)$. The maximum of $\binom{n}{2}$ numbers is at least their average, so $\mathrm{pmax}(z) \ge \frac{n-1}{\binom{n}{2}}\,\mathrm{mass}(z) = \frac{2}{n}\,\mathrm{mass}(z)$, and $\mathrm{mass}(z) > 0$ because some entry is nonzero. $\square$

The bound $2/n$ is deliberately weak. [1] proves the much better $\pi^{\max}_{n,2} \ge \mu^{\max}_{n,2} = 4/(n+1)$ by an appendix argument (its Appendix B) that is not reproduced here; $2/n$ is a cheap independent certificate that the quantity is bounded away from zero, and that is all it is asked to be. The normalisations agree: its Theorem 3.8 reads, for $n \ge 3$ and *every* $z \in \mathbb{C}^n$, $\max_{1\le k<l\le n}(|z_k|+|z_l|+|z_k+z_l|) \ge \frac{4}{n+1}\sum_k |z_k|$, which in the notation (1) is $\mathrm{pmax}(z) \ge \frac{4}{n+1}\,\mathrm{mass}(z)$, i.e. $\delta(z) \ge 4/(n+1)$ verbatim. That it is $\mu^{\max}_{n,2}$ that this value names, and that $\pi^{\max}_{n,2} \ge \mu^{\max}_{n,2}$, are its Remarks 3.6 and 3.4.

4. The regular n-gon is never optimal

[1, Proposition 4.2] records the value of δ at the regular n-gon, $\delta(A_n) = (4/n)\cos^2(\pi/2n)$, and then disposes of it in a clause: “Since $(4/n)\cos^2(\pi/2n) > 4/(n+2/3)$ (similar to an inequality in the proof of Theorem 3.17), the above 3-4 *rectangle* configuration shows that when $d = 2$ and $n \ge 6$ is even, the optimal polygon is never the regular n-gon.” For the pentagon it says only: “We conjecture that for $n = 5$ the optimal polygon is approximately 1.3% better than the regular pentagon.” Neither number is computed there. Both are below. We take the identity $\delta(A_n) = (4/n)\cos^2(\pi/2n)$ from [1] and do not reprove it; what follows is everything downstream of that value.

Theorem 4.1 (the comparison, and the pentagon exactly)**.** (1) *For every* real $x \ge 5$, $\dfrac{4}{x+\frac{2}{3}} < \dfrac{4}{x}\cos^2\!\Big(\dfrac{\pi}{2x}\Big)$.

(2) *Hence for every even* $n = 2m \ge 6$, $\pi^{\max}_{n,2} < \dfrac{4}{n}\cos^2\!\Big(\dfrac{\pi}{2n}\Big) = \delta(A_n)$*: the regular* n*-gon is never optimal.*

(3) *At the pentagon,* $\dfrac{4}{5}\cos^2\!\Big(\dfrac{\pi}{10}\Big) = \dfrac{5+\sqrt{5}}{10}$ *exactly, and* $\pi^{\max}_{5,2} < \dfrac{5+\sqrt{5}}{10}$.

(4) *The house improves on the regular pentagon by exactly*

$$1 - \frac{5/7}{(5+\sqrt{5})/10} = \frac{5\sqrt{5}-11}{14}, \qquad \textit{and} \qquad 0.0128 < \frac{5\sqrt{5}-11}{14} < 0.0129.$$

Proof sketch. (1) is the sketch of [1] made exact. Write $t = \pi/x$. The half-angle identity gives $\cos^2(\pi/2x) = \frac{1}{2} + \frac{1}{2}\cos t$, so the claim is $1 - \cos t < 4/(3x+2)$. Combine $1 - \cos t \le t^2/2$ with $t^2/2 < 4/(3x+2)$, the latter being $\pi^2(3x+2) < 8x^2$, which follows from $\pi < 3.15$ and $x \ge 5$ by a quadratic estimate. (2) is (1) at $x = n$ together with Theorem 3.3. (3) needs no bound on π: the exact value $\cos(\pi/5) = (1+\sqrt{5})/4$ gives $(4/5)\cos^2(\pi/10) = \frac{2}{5}(1+\cos(\pi/5)) = (5+\sqrt{5})/10$, and then $5/7 < (5+\sqrt{5})/10$ is $225 < 245$ after clearing. (4) is elementary algebra in $\sqrt{5}$, and the numerical bracket follows from $2.236 < \sqrt{5} < 2.2361$. $\square$

Part (1) is stated for every real $x \ge 5$ and not merely for even integers $n \ge 6$, and the hypothesis $x \ge 5$ is not decoration: at $x = 4$ the inequality is *false*, because $(4/4)\cos^2(\pi/8) = (2+\sqrt{2})/4 \approx 0.85355$ is below $4/(4+\frac{2}{3}) = 6/7 \approx 0.85714$. That is exactly why [1] states Conjecture 4.4 for $n \ge 6$ and not for $n \ge 4$, and it is recorded as a control in Section 6. Part (4) turns the “approximately 1.3%” of [1] into an algebraic number: $(5\sqrt{5}-11)/14 = 0.012881\ldots$

5. Active pairs, and the answer at $n = 4$

5.1. **Two more configurations.**

Theorem 5.1 (the collapsed pair, and the square)**.** *For $n \geq 2$ let $C_n = (1, -1, 0, \ldots, 0)$, with $n-2$ zero entries. Then C_n is a non-trivial zero-sum configuration — but not an admissible one — with* $\mathrm{pmax}(C_n) = 2$ *and*

$$\mathrm{act}(C_n) = \binom{n}{2} - \binom{n-2}{2} = 2n - 3.$$

Let $S = (1, i, -1, -i)$ be the unit square. Then S is admissible of size 4, $\delta(S) = (2+\sqrt{2})/4$, hence $\pi^{\max}_{4,2} \leq (2+\sqrt{2})/4$, and $\mathrm{act}(S) = 4$ *of its* 6 *pairs.*

Proof sketch. For C_n every pair value is 2 — $\mathrm{pv}(1,-1) = 2$ and $\mathrm{pv}(\pm 1, 0) = 2$ — except the pairs of two zeros, where it is 0; so the maximum is 2 and the active pairs are exactly those meeting a nonzero entry, of which there are $\binom{n}{2} - \binom{n-2}{2} = 2n-3$. The count uses the same two counting lemmas as Theorem 3.4. For S, four of the six pair values are $2+\sqrt{2}$ (the pairs of perpendicular edges) and two are 2 (the pairs of opposite edges), and $\mathrm{mass}(S) = 4$. The value $(2+\sqrt{2})/4$ is the one [1] records for $\pi^{\max}_{4,2}$ (its Corollary 2.7 and Proposition 3.3); here it appears only as an upper bound, the matching lower bound being that paper's and not reproved. □

The configuration C_n is degenerate as a polygon — a doubly traversed segment — and does not appear in [1], which considers only configurations it expects to be optimal. It is admitted here for one reason: it satisfies the definition of *non-trivial* that the open question of [1, Remark 4.5] is posed with, and it lies in the feasible set of [1, Proposition 3.3].

5.2. **The main result.**

Theorem 5.2 (an optimum that is not active-maximal)**.** *At $n = 4$,*

$$\mathrm{act}(S) = 4, \qquad \mathrm{act}(C_4) = 5, \qquad \mathrm{act}(S) < \mathrm{act}(C_4).$$

Consequently, since [1, Theorem 3.16] *at $n = 4$ (where $d = n-2 = 2$) shows that the square is the* unique *optimal configuration for $\pi^{\max}_{4,2}$, an optimal configuration for $\pi^{\max}_{4,2}$ does not achieve the maximal number of simultaneously active pairs over non-trivial zero-sum configurations of size* 4*. The second half of the open question of* [1, Remark 4.5] *has a negative answer at $n = 4$.*

Proof. The two counts are Theorem 5.1 at $n = 4$: $2 \cdot 4 - 3 = 5$ and $\mathrm{act}(S) = 4$. The configuration $C_4 = (1, -1, 0, 0)$ is non-trivial in the sense of [1, §3.1], so the maximum in the question is at least 5, while the optimal configuration attains 4. □

Three things about Theorem 5.2 are worth stating plainly, because each of them is a place where the claim could be read as wider than it is.

First, the *uniqueness* of the square as the optimum at $n = 4$ is quoted from [1] and is not reproved here; what is proved here is the pair of counts and their strict inequality, which is what settles the question once that uniqueness is granted. In fact uniqueness is more than is needed: it suffices that the square be *an* optimum. The statement being used is [1, Theorem 3.16] — "For $n \geq 4$, the vertices of a regular n-gon centered about the origin are the only optimal configurations for $\pi^{\max}_{n,n-2}$" — taken at $n = 4$, where $n - 2 = 2$. Three features of it matter here and all are in our favour. It optimises over the same feasible set $\mathcal{Z}_n$ as [1, Proposition 3.3], so the uniqueness holds over the *non-trivial* class and a fortiori over the admissible one. Its description of the optimum is up to rotation and up to permutation of the entries; at $n = 4$ they are the tuples cS, $c \in \mathbb{C} \setminus \{0\}$, and their reorderings, with the normalisation $\sum_k |z_k| = 1$ pinning $|c| = 1/4$. Since pv is homogeneous and rotation-invariant and act is unchanged by reordering, act is the same at every one of them, namely 4. And it is a statement about π, not about μ: the classification of the optimal configurations for $\mu^{\max}_{n,2}$ is a different theorem of [1] (its Theorem 3.12), and it is not the one used here.

Second, the answer depends on which class *non-trivial* names, and the dependence is not a technicality. Under the definition of [1, §3.1], which excludes only the all-zero configuration — and which is the class over which [1, Proposition 3.3] takes the minimum defining $\pi^{\mathrm{nt}}_{n,2}$ in the first place — the answer at $n = 4$ is *no*, as above. Under the stricter reading in which every z_k is required to be nonzero — what admissibility asks, and what a genuine 4-gon needs — the configuration C_4 is no longer a competitor, and the answer at $n = 4$ turns into *yes*; see Remark 5.5. Both readings are worth having, and only the first is settled by a machine-checked argument.

Third, $n = 4$ is an anomaly and not the beginning of a pattern.

Proposition 5.3 (the phenomenon is confined to $n = 4$)**.**

$$\mathrm{act}(C_4) = 5 > 4 = \mathrm{act}(R_2), \qquad \mathrm{act}(C_5) = 7 = \mathrm{act}(H),$$
$$\mathrm{act}(C_6) = 9 < 10 = \mathrm{act}(R_3), \qquad \mathrm{act}(C_8) = 13 < 18 = \mathrm{act}(R_4).$$

Remark 5.4. Comparing the two closed forms of Theorems 5.1 and 3.4 explains the table: at $n = 2m$ the degenerate count $2n - 3 = 4m - 3$ exceeds $m^2 + m - 2$ exactly when $m^2 - 3m + 1 < 0$, that is only for $m = 2$. This one-line consequence of the two theorems is not itself part of the formal development, which certifies the four numerical comparisons above.

Remark 5.5 (what is known about the first half of the question, and how). The first half of the question of [1, Remark 4.5] — the value of the maximum itself — is not settled here, and the following two records are stated for the reader's orientation only. They are *not* part of the machine-checked development and carry a different standing from every numbered statement above.

- *A hand proof at $n = 4$.* An elementary planar argument, written out but not formalised, gives: over non-trivial zero-sum configurations of size 4 the maximal number of simultaneously active pairs is exactly 5, attained only by $(u, -u, 0, 0)$ up to permutation; over configurations with every $z_k \neq 0$ it is exactly 4, attained by the square and by R_2. The argument runs in two claims. All six pairs active forces $|z_1| = |z_2| = |z_3| = |z_4|$ and then all six pairwise angle cosines equal to $-1/3$, so the imaginary parts of three unit vectors, each $\pm 2\sqrt{2}/3$, would have to sum to zero. Exactly five active (inactive pair $\{3, 4\}$) forces $|z_1| = |z_2| = s$, $|z_3| = |z_4| = t$ with $t < s$ and $|z_1 + z_3| = |z_1 + z_4|$, hence $\mathrm{Re}(\bar{z}_1(z_3 - z_4)) = \mathrm{Re}(\bar{z}_2(z_3 - z_4)) = 0$; both branches collapse to $z_3 = z_4 = 0$, $z_2 = -z_1$. This has been read once by its author and is corroborated numerically; it has not been formalised or refereed, and we make no stronger claim for it than that.
- *A floating-point feasibility search.* For each target set T of pairs, a least-squares system asking $\mathrm{pv}_{ij} = M$ for $(i, j) \in T$ together with $\sum_k z_k = 0$, $\sum_k |z_k| = 1$ and one-sided penalties off T was solved from 12 random starts, a residual below 10^{-20} counting as achievable; about 50 CPU-minutes in total. At $n = 4$, 6 pairs were never achieved and 5 only with two vanishing entries; at $n = 5$, 10, 9 and 8 were never achieved over all 1, 10 and 45 target sets, and 7 was; at $n = 6$, $15, 14, 13, 12, 11$ were never achieved over all 1, 15, 105, 455 and 1365 target sets, and 10 was — at a configuration the search found on its own with moduli in ratio 4:3:3:3:3:4, which is the 3-4 rectangle. Every negative here means "not found by this search", never "impossible"; a target set is declared unachievable when 12 restarts fail, which can be a failure of the search rather than an obstruction. Only the $n = 4$ negatives have a proof behind them.

Taken together these say that the expectation of [1] — that an optimal configuration activates the maximum number of pairs — appears to hold at $n = 5$ and $n = 6$, and that $n = 4$ is the exception. Theorem 5.2 is the part of that picture that is proved.

6. Controls

Every count and every value above is a statement that a badly stated definition could satisfy vacuously, and every side condition above is one that a reader is entitled to suspect of being decoration. The following twelve checks were computed from the same definitions as the results they guard.

Proposition 6.1 (negative controls)**.** (1) Too small, too large. *$\delta(H) \not\lesssim 0.714$ and $\delta(H) \not\gtrsim 0.715$; $\mathrm{pmax}(H) \not\lesssim 79$; $\mathrm{act}(H) \notin \{6, 8, 10\}$; $\mathrm{pmax}(R_3) \not\lesssim 11$ and $\mathrm{pmax}(R_3) \not\gtrsim 13$; $\mathrm{act}(R_3) \notin \{9, 11\}$, out of* 15 *pairs.*

(2) The hypothesis $m \geq 2$ in Theorems 3.3 and 3.4. *At $m = 1$ the configuration degenerates to $(4i, -4i)$, whose maximum is 8 and not 12, and whose active count is 1 while $m^2 + m - 2 = 0$.*
(3) The hypothesis $x \geq 5$ in Theorem 4.1(1). *At $x = 4$ the inequality is false:* $(4/4)\cos^2(\pi/8) = (2 + \sqrt{2})/4 < 6/7 = 4/(4 + \frac{2}{3})$.
(4) The clauses of admissibility. *The nonzero clause is not vacuous: C_4 is non-trivial but not admissible. The zero-sum clause is not vacuous: $(1, 1)$ fails it. And admissibility is satisfiable with positive mass: H is admissible of size 5 with $\mathrm{mass}(H) = 112 > 0$, so the upper-bound principle used throughout is not applied to an empty hypothesis.*

Item (3) is the sharpest of the four. It is not merely that the proof of Theorem 4.1(1) needs $x \geq 5$; the statement itself fails at $x = 4$, which is why Conjecture 4.4 of [1] begins at $n = 6$, and it means the inequality cannot be extended downward by any argument.

7. What is not settled, and why

Optimality. Nothing above touches the optimality halves of Conjectures 4.3 and 4.4 of [1] — that 5/7 is the *minimum* over pentagons and $4/(n + \frac{2}{3})$ over even n. In the pentagon case the missing statement is

$$\forall z \in \mathbb{C}^5 : \quad \sum_k z_k = 0 \ \wedge \ \sum_k |z_k| = 1 \implies \max_{1 \leq i < j \leq 5} \big(|z_i| + |z_j| + |z_i + z_j|\big) \ \geq \ \tfrac{5}{7}, \tag{3}$$

and the shape of (3) is what blocks it. The conclusion is a *disjunction* over the ten pairs: for every z, *some* pair is large. That is a covering argument, not a single positivity certificate, so the box-based positivity checkers that settle "this one polynomial is nonnegative on this box" do not apply — here no single polynomial is. Clearing the square roots by introducing r_k with $r_k^2 = x_k^2 + y_k^2$ turns (3) into a Positivstellensatz problem in 15 variables over a semialgebraic set of dimension 6 (10 real coordinates, less 2 for the sum, 1 for scaling and 1 for rotation). The route that would work is an exact *rational* sums-of-squares certificate, refined per pair-region, together with a checker that verifies such a certificate in exact arithmetic; we do not have one, and building it is a separate piece of work with its own correctness burden. [1] itself claims only to have "verified that the house is a strict local minimum up to rotation and scaling".

We record the price honestly, because "not attempted" and "attempted and too expensive" are different states. *No computation at all was spent on the optimality direction*: the obstacle is the absence of an instrument, not a budget that ran out. What the results above did cost is small and is stated in Section 9.

Other open ends.

- *The maximum active count at $n = 5$ and $n = 6$.* Remark 5.5 reports 7 and 10 as the numerical maxima, attained by the house and by R_3; proving either would settle the first half of the question of [1, Remark 4.5] at that n. The $n = 4$ argument does not transfer verbatim, because for $n \geq 5$ the zero-sum relation no longer pairs the indices two by two.
- *Odd n.* [1] reports that its experiments suggest $\pi^{\max}_{n,2}$ is asymptotic to $4/(n + \frac{5}{9})$ for odd $n \geq 7$, and exhibits no configuration. Finding an exact configuration at $n = 7$ that beats the regular heptagon is a target of exactly the shape of Theorems 3.1 and 3.3; nothing here attempts it.
- *A general upper bound on the active count.* All the data above — $2n - 3$, $m^2 + m - 2$, and the numerics — lie strictly below $\binom{n}{2}$ for $n \geq 4$, and no configuration with *all* pairs active is known for $n \geq 4$. (At $n = 3$ all three pairs are always active, since $z_3 = -(z_1 + z_2)$ makes every pair value equal to $\mathrm{mass}(z)$; this is elementary and is not part of the formal development.) Whether $\binom{n}{2}$ is ever attained for $n \geq 4$ is open; the case $n = 4$ is settled negatively by the hand argument of Remark 5.5.

8. Provenance

Only Theorem 5.2 and Proposition 5.3 are offered as new. Theorems 3.1–3.4 certify values and counts that [1] prints; Theorem 4.1 makes exact two assertions that [1] makes without computing; Theorem 3.5

is an averaging bound, weaker than that paper's own; and Theorem 5.1 is an elementary computation, separated from Theorem 5.2 because it is what the latter is for. Before that separation was made, all of the values [1] prints — the perimeter 112, the ratio 5/7, the count 7 of 10, $\delta(R_m) = 4/(n + \frac{2}{3})$ and the count m^2+m-2 for $m = 2, \ldots, 7$, $\delta(A_4) = (2+\sqrt{2})/4$, and the 1.3% — were reproduced independently in exact rational arithmetic, and all agree. The one place where the reproduction *disagrees* with a printed assertion is instructive and is recorded as Proposition 6.1(3): the clause $(4/n)\cos^2(\pi/2n) > 4/(n+2/3)$ is false at $n = 4$, which is consistent with [1] restricting Conjecture 4.4 to $n \geq 6$.

For Theorem 5.2 the following was searched, on 2026-08-30. The source is at version 1, posted fifteen days earlier, and has no citations recorded on either Semantic Scholar (paper identifier `d87440aa…`) or OpenAlex (`W7203660346`). In a local full-text corpus of arXiv mathematics preprints with coverage through the 2026-08-28 announcement, the discriminating string "3-4 rectangle" occurs only inside the source itself, and the identifier 2608.15444 occurs nowhere as a citation. The nearest relative, Nesterenko [2] — which [1] credits with the same complex-squaring reduction — was fetched and read in full: its functional is $\max_{i,j}\bigl(|a_i| + |a_j| - |a_i + a_j|\bigr)$, with a *minus* sign — the combination that occurs in the $d = n - 2$ case of [1], not the $d = 2$ functional (1); it contains no pentagon, no value 5/7 and no active-pair count. The "small polygons" line [3], which [1] cites as structurally related, was *not* read: its functional is a diameter-constrained perimeter or area rather than the ratio δ, and its optimal pentagons are not lattice polygons. Finally, the active counts $4, 10, 18, 28, 40, 54, 70$ of $R_2, \ldots, R_8$ match OEIS A028552, $n(n + 3)$, under a shift; that is the same quadratic and carries no apparent connection to this problem, and is recorded only so that a later reader does not have to rediscover the coincidence.

The channels *not* searched for prior art, which bounds the claim: MathSciNet, zbMATH, Google Scholar, and any publication absent from arXiv and from the corpus above. "Not found" is not "new". What weakens the claim most is simply that [1] is fifteen days old, so the relevant literature has had no time to appear.

Every section and result number attributed to [1] above was resolved by compiling the version 1 e-print and reading the numbers off the compiled document, not by counting sections in the source text: the paper has four numbered sections and two appendices, its §4 is "Conjectures and Open Problems", and the closing open question is its Remark 4.5. The arXiv abstract pages of [1] and [2] were re-read on 2026-08-30; both list version 1 as their only version and neither carries a journal reference.

9. Verification

Every theorem and proposition above has been formally verified in Lean 4 with Mathlib, in the directory `LeanProblemSpec/ActivePairs/` of a repository that is private at the time of writing (repository reference to be added at submission); the development is six modules and 46 headline declarations, and contains no `sorry` and no declared axiom. The exceptions are the two records collected in Remark 5.5 and the one-line observations of Remark 5.4 and of Section 7, each of which says at the point where it stands that it is outside the formal development. Two features of that development are worth naming because they are unusual. *No statement here rests on an exhaustive computation*: there is no enumeration, no search, and no appeal to compiled evaluation — Lean's `native_decide` is not used anywhere, and all 46 declarations carry exactly the axiom set {`propext`, `Classical.choice`, `Quot.sound`}, with `Classical.choice` entering only through the real numbers (their order and the infimum (2)) and through the classical decidability of equality inside the active-pair count. And *no floating-point arithmetic enters at any point*: the two configurations of [1] are lattice configurations whose side and diagonal lengths are integers, so every value of pv in Sections 3 and 5 is an integer certified by three Pythagorean identities over $\mathbb{Z}$, and the four irrational quantities that do occur — $\sqrt{241}$, $\sqrt{2}$, $\sqrt{5}$ and π — are each handled by one explicit rational bound ($\sqrt{241} < 39$; $\sqrt{2} < 1.415$; $2.236 < \sqrt{5} < 2.2361$; $\pi < 3.15$) or by an exact identity ($\cos(\pi/5) = (1 + \sqrt{5})/4$). Counting arguments that a reader might expect to be done n by n — the $m^2 + m - 2$ of Theorem 3.4 and the $2n - 3$ of Theorem 5.1 — are proved uniformly, from two general lemmas about counting over the unordered pairs of a list. The finite searches that were run are all outside the formal development and all reported as such: the exact-rational reproduction of the printed values of [1] described in Section 8, and the floating-point feasibility search of Remark 5.5, whose negatives are not proofs. The cost of the formal part, measured end to end in import order, is

11.6, 10.6, 10.2, 60.2, 177.5 and 12.9 seconds for the six modules — under five minutes of processor time in total — with peak resident sets between 6.80 and 7.05 GB against a 6.88 GB baseline for importing the library alone, so the marginal memory cost is essentially zero. All of it builds under the Lean toolchain `leanprover/lean4:v4.32.0` with Mathlib pinned at its `v4.32.0` tag.

References

[1] J. Acevedo, G. Blekherman, S. Debus, S. Lee and C. Riener, *Eigenvalues of locally positive semidefinite matrices: Non-convexity and Geometry*, arXiv:2608.15444 (15 August 2026). Primary arXiv category `math.RA`, cross-listed `math.AG` and `math.OC`; MSC 2020: primary 15A99, secondary 14P10, 90C47. All quotations and all result numbers here are from version 1, read from the LaTeX e-print and its compiled form on 2026-08-30: §4 was read in full, as were §3.1 and Theorems 3.8, 3.12, 3.16 and 3.17, which are cited but not reproved, and Figures 3a and 3b, which were read from the e-print's vector graphics. Still at version 1 and carrying no journal reference on its arXiv abstract page when re-checked on 2026-08-30.

[2] Yu. Nesterenko, *Submatrices with the best-bounded inverses: Studying* $\mathbb{R}^{n\times 2}$ *and* $\mathbb{C}^{n\times 2}$, arXiv:2408.16631 (2024). Cited by [1] for the same complex-squaring reduction; read in full here. Its functional is $\max_{i,j}(|a_i|+|a_j|-|a_i+a_j|)$, not the functional of this paper. Preprint only: no journal version is recorded on its arXiv abstract page, in Crossref or in dblp, all checked 2026-08-30.

[3] D. Henrion and F. Messine, *Finding largest small polygons with GloptiPoly*, J. Global Optim. 56 (2013), no. 3, 1017–1028; C. Bingane and M. J. Mossinghoff, *Small polygons with large area*, J. Global Optim. 88 (2024), no. 4, 1035–1050; B. Mulansky and A. Potschka, *A zonogon approach for computing small convex polygons of maximum perimeter*, Math. Program. 218 (2026), no. 1–2, 563–589, with a correction, ibid., 593–596. Cited from [1] as the structurally related "small polygons" line; not read here. Their functionals are diameter-constrained perimeter or area rather than the ratio δ of (1), and their optimal pentagons are not lattice polygons. Volumes, issues, pages and the year of the printed issue are from the publishers' Crossref records (`doi:10.1007/s10898-011-9818-7`, `doi:10.1007/s10898-023-01329-1`, `doi:10.1007/s10107-025-02244-x` and `doi:10.1007/s10107-025-02257-6`), corroborated for the last two by dblp, and not from the bibliography of [1], which cites the third under its preprint title and as "online first".

[4] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`, entry A028552 ($n(n+3)$), read 2026-08-30. Cited in Section 8 only to record a coincidence of quadratics, not a connection.

[5] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381, `doi:10.1145/3372885.3373824`. The version used here is the `v4.32.0` tag.